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Quantizing non-Lagrangian gauge theories: an augmentation method

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Abstract: We discuss a recently proposed method of quantizing general non-Lagrangian gauge theories. The method can be implemented in many different ways, in particular, it can employ a conversion procedure that turns an original non-Lagrangian field theory in d dimensions into an equivalent Lagrangian, topological field theory in $d+1$ dimensions. The method involves, besides the classical equations of motion, one more geometric ingredient called the Lagrange anchor. Different Lagrange anchors result in different quantizations of one and the same classical theory. Given the classical equations of motion and Lagrange anchor as input data, a new procedure, called the augmentation, is proposed to quantize non-Lagrangian dynamics. Within the augmentation procedure, the originally non-Lagrangian theory is absorbed by a wider Lagrangian theory on the same space-time manifold. The augmented theory is not generally equivalent to the original one as it has more physical degrees of freedom than the original theory. However, the extra degrees of freedom are factorized out in a certain regular way both at classical and quantum levels. The general techniques are exemplified by quantizing two non-Lagrangian models of physical interest.

Keywords: BRST Symmetry, BRST Quantization, Topological Field Theories.

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Contents

1. Introduction

Classical dynamics can be consistently formulated in terms of equations of motion alone. The variational principle, being a useful tool for studying various aspects of classical dynamics, is not needed to have the classical theory defined as such. However, to promote the classical dynamics to quantum level, it is insufficient to know only the equations of motion, one or another extra structure is needed. If the quantum theory is supposed to be formulated in the language of Feynman's path integrals, it is the action functional that can serve as the additional ingredient needed for quantization. On the other hand, any

Lagrangian equations of motion can always be brought to a (constrained) Hamiltonian form that makes possible applying canonical quantization. Furthermore, the method of deformation quantization applies to the Hamiltonian systems even though the underlying Poisson bracket is degenerate [1] (in which case the Hamilton equations can have no variational formulation). As it has been recently found [2, 3], the deformation quantization can also be implemented under far less restrictive conditions on the equations of motion than the requirement to be Hamiltonian. Roughly speaking, the phase-space evolution flow is not required to be Hamiltonian: It is sufficient if the evolution preserves the Poisson bracket modulo constraints and gauge transformations. The bracket, in its turn, is also required to satisfy the Jacobi identity in a weak sense, i.e., modulo constraints and gauge transformations. For accurate definitions, see [2, 3].

So, the deformation quantization has progressed in recent years reaching far beyond the range of theories admitting variational principle for equations of motion. At the same time, the methods of constructing the partition functions[1] dating back to Feynman, Schwinger and Dyson, and being now developed in full generality for arbitrary Lagrangian gauge theories [4 − 6], have not made much progress in the class of theories having no action functional. Until recently, no general method has been known to path-integral quantize a non-Lagrangian theory as it was not clear what might be a generalization of the familiar Schwinger-Dyson's equation in the situation where no Lagrangian formulation is possible for the classical equations of motion.

In our recent papers [7, 8], we have identified a general structure, called the Lagrange anchor, which is determinative for the quantization in terms of partition functions in the same sense as the Poisson bracket defines deformation quantization in terms of a star product. The Lagrange anchor is a geometric object that can be interpreted in many different ways. In particular, one could say that the Lagrange anchor is related to the canonical anti-bracket of the Batalin-Vilkovisky formalism [5] much like a generic (i.e., possibly degenerate and non-constant rank) Poisson bracket is related to the canonical Poisson bracket. The anchor is also required to satisfy certain compatibility conditions involving equations of motion. In Lagrangian theory, these conditions are automatically satisfied for the canonical anti-bracket in consequence of the fact that the equations of motion are variations of the action functional. If the anchor is invertible, these compatibility conditions ensure existence of the equivalent Lagrangian formulation. It turns out that the partition function can still be constructed by making use only of the equations of motion and the Lagrange anchor, even though the latter is degenerate, defining no action functional. The next section contains an accurate definition of the Lagrange anchor and discussion of its properties.

By now two procedures have been worked out to construct partition functions for general non-Lagrangian theories. The first one [7] suggests a conversion of the original non-Lagrangian field theory in d dimensions into an equivalent $(d + 1)$-dimensional Lagrangian topological theory which can then be quantized by the standard BV method. The conversion procedure is quite ambiguous and essentially depends on the choice of the

[1]In the following we will also use the term *probability amplitude* as a synonym for the *partition function.*

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Lagrange anchor. If the anchor was invertible (that assumes implicitly the existence of some action for the original dynamics), the path integral can be explicitly taken in the bulk of the topological theory resulting in Feynman's partition function for original action. With a general (non-invertible) anchor, the answer for the partition function cannot be reduced to the canonical form, but it remains fully consistent and allows quite natural physical interpretation [8]. If the anchor is chosen to be zero, the partition function will correspond to the classical transition amplitude [9]. The second method to quantize a classical theory with non-Lagrangian equations of motion [8] suggests a nontrivial generalization of the Schwinger-Dyson equation that any partition function must satisfy. This equation, involving classical equations of motion and the Lagrange anchor, reduces to the BV quantum master equation whenever the anchor is invertible.

In this paper, we propose an alternative procedure of constructing partition functions for general dynamical systems. This procedure starts with the same input data: classical equations of motion and the Lagrange anchor, but it exploits quite different idea and technology. We call this procedure an *augmentation* because it is motivated by a widespread view that either a non-Lagrangian system can be reshaped into an equivalent Lagrangian model in an appropriately extended configuration space, or it describes an effective dynamics emerging from a Lagrangian theory after averaging over some degrees of freedom or their exclusion from the equations of motion. So, the intuitive intention about quantizing a non-Lagrangian theory is to augment it first to a Lagrangian one, and then the augmented theory can be quantized in the usual way. No general method is known to date to *equivalently* reformulate any given non-Lagrangian model as a Lagrangian one by adding a finite number of new fields. We propose a uniform procedure to construct an augmented Lagrangian theory for any (non-)Lagrangian dynamics, which is not however an equivalent reformulation. The augmented theory may have, in principle, more degrees of freedom than the original model, but classically, the original dynamics are easily singled out by imposing appropriate boundary conditions on the extra fields. These boundary conditions guarantee that the original fields evolve precisely in the same way as in the original theory, while new fields do not evolve at all. This reduction mechanism always restores the original dynamics in the augmented theory including the case where the original theory is Lagrangian. Quantizing the augmented Lagrangian system by conventional BV procedure and integrating the new fields out in the path integral, one gets the original (not necessarily Lagrangian) dynamics quantized. If the original theory is Lagrangian, the integral can be taken explicitly over the augmentation fields with corresponding boundary conditions, and the partition function obtained in this way will coincide with that constructed from the BV master action for the original Lagrangian. If the original theory is not Lagrangian, the constructed partition function is still correct that can be seen in several ways, although it cannot be represented anymore as an exponential of any (local or non-local) action functional.

Let us also comment on an essential distinction between the augmentation idea we use to quantize non-Lagrangian theories and somewhat similar concept of "auxiliary fields" [10]. The fields are usually understood as auxiliary when they are introduced to extend the dynamics in such a way that the extended classical theory remains fully equivalent to the

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original one. In particular this means that the number of independent initial data for Cauchy problem remains the same as in the original theory. In contrast to introducing the auxiliary fields, the augmentation procedure results in a theory that has more degrees of freedom than the original one. The extra dynamics are eliminated, however, by imposing zero initial and/or boundary conditions on the augmentation fields. At quantum level, these conditions provide the absence of the "augmenting particles" in in- and out-states of the quantum system. It might be relevant to mention that no regular procedure is known yet for introducing auxiliary fields in such a way as to convert any non-Lagrangian theory into an equivalent Lagrangian one. In some specific models the way of introducing auxiliary fields is known, although it often happens that the restrictions are to be imposed strongly limiting the admissible form of equations of motion.[2] In contrast, the augmentation is a regular procedure which always works well, given equations of motion and Lagrange anchor.

The paper is organized as follows. To make the paper self-contained, we review some recent developments in path-integral quantization of non-Lagrangian theories that includes basic definitions and some relevant statements from [7, 8]. In section 2, we set up notation and explain some basic facts concerning the general structure of not necessarily Lagrangian gauge systems. We recall the notion of Lagrange structure, which contains the Lagrange anchor as a key ingredient, and put it in the context of S_∞-algebras. The corresponding Subsection 2.4 is addressed to the readership familiar with basics of strongly homotopy algebras, others may just omit this subsection. The paper can be further understood without knowing the concept of S_∞-algebras, although this concept provides a natural homological insight into the quantization problem of non-Lagrangian dynamics. In section 3, we describe a BRST complex, which can be assigned to any (non-)Lagrangian gauge system. As input data, this complex involves the original equations of motion, generators of gauge identities and gauge symmetries, and the Lagrange anchor. At first, we define the ambient Poisson manifold that hosts this BRST complex and construct the BRST charge by homological perturbation theory. Further, we show that the BRST cohomology classes precisely correspond to the physical observables of the original (non-)Lagrangian theory. We also give an important interpretation of this BRST complex as that resulting from the BFV-BRST quantization of some constrained Hamiltonian system on the phase space of original fields and their sources. Section 4 is devoted to the quantization of the BRST complex. Quantizing the ambient Poisson manifold, we define the quantum BRST cohomology and present the generalized Schwinger-Dyson equation for the partition function of a (non-)Lagrangian gauge theory. This equation is shown to have a unique solution, which can be written down in a closed path-integral form.

Section 5 contains the main results of the paper. Namely, in section 5.1. we define the augmented BRST complex, which is build on the original BRST complex and carries all the information about the augmented theory. In section 5.2. we unfold the structure

[2] For example, in higher-spin field theories, the auxiliary fields can be introduced converting a non-Lagrangian model into Lagrangian one unless no interaction has been switched on, even though the consistent equations of motion with interaction are know for many years, see [11] for a review and further references

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of the augmented BRST charge by interpreting it in terms of equations of motion, gauge symmetry and Noether identity generators. As the Lagrange anchor behind the augmented theory is always nondegenerate (whatever the original anchor), the augmented partition function has the standard Feynman's form. Moreover, the corresponding action functional is proved to possess the property of space-time locality provided the original equations of motion and the Lagrange anchor do so. Finally, in section 5.3. we present an alternative path-integral representation for the quantum averages of the original physical observables in terms of the augmented action functional.

In section 6, we apply the augmentation procedure to quantize two non-Lagrangian field theories: Maxwell electrodynamics with monopoles and self-dual p-forms. These models are known to admit no Lagrangian formulation. However, we have found non-trivial (degenerate) Lagrange anchors for these theories. Making use of the anchors, we apply the augmentation method to construct manifestly Poincaré invariant partition functions for both the models.

2. Lagrange structure and S_∞-algebras

2.1 Classical dynamics

In field theory one usually deals with the space Y^X of all smooth maps from a space-time manifold X to a target manifold Y. The atlases of coordinate charts on X and Y define then a natural atlas on Y^X such that each map $x : X \to Y$ is specified locally by a set of smooth fields x^i, the coordinates on the infinite-dimensional manifold Y^X. Hereafter we use De Witt's condensed notation [4], whereby the superindex "i" comprises both the local coordinates on X and the discrete indices labelling the components of the field x. As usual, the superindex repeated implies summation over the discrete indices and integration over the space-time coordinates w.r.t. an appropriate measure on X. The partial derivatives $\partial_i = \partial/\partial x^i$ are understood as variational ones.

In the context of local field theory, the space Y^X is known as the *space of all histories* and the *true histories* are specified by a set of PDE's

$$T_a(x) = 0\,. \tag{2.1}$$

Here we do not assume the field equations to come from the least action principle, hence the indices i and a, labelling the fields and equations, may run through completely different sets. In the case where X is a manifold with boundary, eqs. (2.1) are also supplemented with a suitable set of boundary conditions. Usually, the boundary conditions specify the values of fields and/or their derivatives up to some fixed order. Varying these values, collectively called the *boundary data*, one gets a family of different solutions to eqs. (2.1).

For our purposes it is convenient to think of $T = \{T_a(x)\}$ as a section of some vector bundle $\mathcal{E} \to M$ over subspace of all fields $M \subset Y^X$ with given boundary data. Then the set of all true histories Σ belonging to M is identified with zero locus of $T \in \Gamma(\mathcal{E})$:

$$\Sigma = \{x \in M|\; T(x) = 0\;\}\,. \tag{2.2}$$

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Using the physical terminology, we refer to Σ as the *shell*. Under the standard regularity conditions [6], $\Sigma \subset M$ is a smooth submanifold associated with an orbit of gauge symmetry transformations (see eq. (2.9) below); in the absence of gauge symmetries the shell Σ is a single point of M. In the following we will always assume Σ to be a *connected* submanifold for each choice of boundary data.

Thus, the classical dynamics are completely specified by a section T of some vector bundle $\mathcal{E} \to M$ over the space of all histories subject to boundary conditions. For this reason we call $\mathcal{E}$ the *dynamics bundle*.

2.2 Regularity conditions

To avoid pathological examples, some regularity conditions are usually imposed on a classical system. To formulate these conditions in an explicitly covariant way, let us introduce an arbitrary connection ∇ on $\mathcal{E}$ and define the section

$$J = \nabla T \in \Gamma(T^*M \otimes \mathcal{E}) \tag{2.3}$$

This section, in turn, defines the M-bundle morphism [3]

$$J : TM \to \mathcal{E}\,, \tag{2.4}$$

which is not necessarily of constant rank.

Definition 1. *A classical system* $(\mathcal{E}, T)$ *is said to be* regular of type (m,n)*, if there exists a finite sequence of vector bundles* $\mathcal{E}_k \to M$ *and* M*-bundle morphisms*

$$0 \to \mathcal{E}_{-m} \to \cdots \to \mathcal{E}_{-1} \xrightarrow{R} TM \xrightarrow{J} \mathcal{E} \xrightarrow{Z} \mathcal{E}_1 \to \cdots \to \mathcal{E}_n \to 0 \tag{2.5}$$

satisfying conditions:

(a) there is a tubular neighbourhood $U \subset M$ *of* Σ *such that all the morphisms (2.5) have constant ranks over* U*;*

(b) upon restriction to Σ*, the chain (2.5) makes an exact sequence.*

This definition has several important corollaries elucidating its meaning:

Corollary 1. *The shell* $\Sigma \subset M$ *is a smooth submanifold with* $T\Sigma = \mathrm{Im}\, R|_\Sigma$*.*

Corollary 2. *For any vector bundle* $\mathcal{V} \to M$ *and a section* $K \in \Gamma(\mathcal{V})$ *vanishing on* $\Sigma \subset M$*, there is a smooth section* $W \in \Gamma(\mathcal{E}^* \otimes \mathcal{V})$ *such that*

$$K = \langle T, W \rangle\,, \tag{2.6}$$

where the triangle brackets denote contraction of T *and* W*. Informally speaking, any on-shell vanishing section is proportional to* T*.*

[3] To simplify notation, we will not distinguish between an M-bundle morphism $H : \mathcal{E} \to \mathcal{E}'$, the induced homomorphism $\Gamma(H) : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E}')$ on sections, and the associated section $\widetilde{H} \in \Gamma(\mathcal{E}^* \otimes \mathcal{E}') \simeq \mathrm{Mor}(\mathcal{E}, \mathcal{E}')$, denoting all these maps by one and the same letter H.

Corollary 3. *When exist, the morphisms (2.5) are not unique off shell. Thinking of these morphisms as the sections of the corresponding vector bundles, one can add to them any sections vanishing on* Σ*, leaving the properties (a),(b) unaffected. In particular, by making a shift*

$$Z \to Z + Z_0\,, \qquad Z_0|_\Sigma = 0\,, \tag{2.7}$$

if necessary, we can always assume that $T \in \ker Z$*. In view of the previous remark, the section* Z_0 *is proportional to* T*.*

Corollary 4. *In the definition above we can pass from the sequence (2.5) to the transpose one by replacing each vector bundle with its dual and inverting all the arrows. The transpose sequence will meet the same conditions (a),(b) as the original one.*

In this paper we deal mostly with the quantization of regular $(1,1)$-type Lagrange structures associated to the four-term sequences

$$0 \to \mathcal{F} \xrightarrow{R} TM \xrightarrow{J} \mathcal{E} \xrightarrow{Z} \mathcal{G} \to 0\,. \tag{2.8}$$

The on-shell exactness at TM suggests that for any vector field $V \in \Gamma(TM)$ obeying condition $\nabla_V T|_\Sigma = 0$ there exists a section $\varepsilon \in \Gamma(\mathcal{F})$ such that $V = R(\varepsilon)$. Combining this with Corollary 2, we can write

$$R^i_\alpha \nabla_i T_a = U^b_{\alpha a} T_b \tag{2.9}$$

for some $U \in \Gamma(\mathcal{E}\otimes\mathcal{E}^*\otimes\mathcal{F})$. Here indices a, i, α label the components of the corresponding sections w.r.t. to some frames $\{e^\alpha\} \in \Gamma(\mathcal{F}|_U)$, $\{e^a\} \in \Gamma(\mathcal{E}|_U)$, and $\{\partial_i\} \in \Gamma(TU)$ associated with a trivializing coordinate chart $U \subset M$. Let $\{e^A\}$ be a frame in $\mathcal{G}$ over U. In view of Corollary 3 the on-shell exactness at term $\mathcal{E}$ implies then

$$Z^a_A T_a = 0\,, \tag{2.10}$$

if Z was chosen in an appropriate way. Relations (2.9) and (2.10) have a straightforward interpretation in terms of constrained dynamics [6]: the homomorphism R is identified with an irreducible set of gauge symmetry generators for the classical equations of motion $T = 0$, while the homomorphism Z generates a set of independent Noether's identities. Having in mind this interpretation, we term $\mathcal{F}$ and $\mathcal{G}$ the *gauge algebra bundle* and the *Noether identity bundle*, respectively. Notice that the irreducibility of the gauge symmetry generators is provided by the on-shell exactness of (2.8) at $\mathcal{F}$, while irreducibility of Noether's identity generators follows from the on-shell exactness of the transpose of (2.8) at $\mathcal{G}^*$.

This interpretation of homomorphisms R and Z applies to the general regular systems of type (m,n), except that the bases of the gauge algebra and Noether's identity generators may be overcomplete (reducible). A general $(n+1,m+1)$-type gauge theory with $n>0$ and/or $m>0$ corresponds to the case of n-times reducible generators of gauge transformations and/or m-times reducible generators of Noether's identities. The theories of type $(0,0)$ are described by linearly independent equations of motion having a unique solution.

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2.3 Lagrange structure

In the context of covariant path-integral quantization, the passage from classical to quantum theory involves, besides classical equations of motion, one more geometric ingredient called the Lagrange structure [7].

Definition 2. *Given a classical system* $(\mathcal{E}, T)$*, a* Lagrange structure *is an* $\mathbb{R}$*-linear map* $d_{\mathcal{E}} : \Gamma(\wedge^n \mathcal{E}) \to \Gamma(\wedge^{n+1}\mathcal{E})$ *obeying two conditions:*

(i) $d_{\mathcal{E}}$ *is a derivation of degree 1, i.e.,*

$$d_{\mathcal{E}}(A \wedge B) = d_{\mathcal{E}}A \wedge B + (-1)^n A \wedge d_{\mathcal{E}}B\,,$$

for any $A \in \Gamma(\wedge^n \mathcal{E})$ *and* $B \in \Gamma(\wedge^\bullet \mathcal{E})$*;*

(ii) $d_{\mathcal{E}}T = 0\,.$

Here we identify $\Gamma(\wedge^0\mathcal{E})$ *with* $C^\infty(M)$*.*

Due to the Leibnitz rule (i), in each trivializing chart $U \subset M$ the operator $d_{\mathcal{E}}$ is completely specified by its action on the coordinate functions x^i and the basis sections e^a of $\mathcal{E}|_U$:

$$d_{\mathcal{E}}x^i = V_a^i(x)e^a\,, \qquad d_{\mathcal{E}}e^a = \frac{1}{2}C^a_{bc}(x)e^b \wedge e^c\,. \tag{2.11}$$

Applying $d_{\mathcal{E}}$ to the section $T = T_a e^a$, one can see that the property (ii) is equivalent to the following structure relations:

$$d_{\mathcal{E}}T = (V_a^i \partial_i T_b - C^c_{ab}T_c)e^a \wedge e^b = 0\,. \tag{2.12}$$

The first relation in (2.11) means also that $d_{\mathcal{E}}$ defines a bundle homomorphism $V : \mathcal{E}^* \to TM$. The section $V \in \Gamma(\mathcal{E} \otimes TM)$ is called the *Lagrange anchor*.

Definition 3. *A Lagrange structure* $(\mathcal{E}, T, d_{\mathcal{E}})$ *is said to be* regular *at* $p \in M$*, if there exists a vicinity* $U \subset M$ *of* p *such that the* M*-bundle morphism*

$$R \oplus V : \mathcal{E}_{-1} \oplus \mathcal{E}^* \to TM \tag{2.13}$$

has a constant rank[4] r *over* U*. The number* r *is called the* rank of Lagrange structure at $p \in M$*. The Lagrange structure is said to be* complete *at* p*, if the homomorphism (2.13) is surjective on* U*. Finally, we say that the Lagrange structure is regular (or complete), if it is regular (or complete) at any point of* M*.*

Remark 1. *In view of Definition 1, the regularity of the Lagrange structure at* $p \in \Sigma$ *is equivalent to the regularity at* p *of the anchor morphism* $V : \mathcal{E}^* \to TM$*, i.e., there exists a sufficiently small vicinity* $U \subset M$ *of* $p \in \Sigma$ *such that* V *has constant rank over* U*.*

[4] Of course, in the context of infinite-dimensional manifolds the notion of rank needs clarification. An appropriate definition can be done, for example, in the case of local field theories.

Remark 2. *In the context of quasiclassical quantization we will deal with in sequel, it is also appropriate to introduce the notions of* weakly regular *and* weakly complete Lagrange structures *by requiring regularity and completeness only for the points of* Σ.

Theorem 1 (Splitting theorem [7]) *Let* $p \in M$ *be a regular point of the Lagrange structure* $(\mathcal{E}, T, d_{\mathcal{E}})$*, then there is a coordinate system* $(y^1, \dots, y^r, z^1, \dots, z^k)$ *centered at* p *together with a set of local functions* $S(y)$*,* $E^1(y), \dots, E^k(y)$ *such that equations* $T_a(y,z) = 0$ *are equivalent to*

$$\frac{\partial S(y)}{\partial y^I} = 0\,, \qquad z^J = E^J(y)\,,$$

and the Lagrange anchor $V = (V^J, V_I)$ *is given by the abelian vector distribution*

$$V^J = 0\,, \qquad V_I = \frac{\partial}{\partial y^I} + \frac{\partial E^J}{\partial y^I}\frac{\partial}{\partial z^J}\,.$$

Here the number r *is the rank of the Lagrange structure at* $p \in M$.

In case $r < \dim M$, it is natural to call $S(y)$ a *partial action.*

Although the theorem above ensures the split of local coordinates into "Lagrangian" y's and "non-Lagrangian" z's, it is by no means necessary to explicitly perform this splitting in order to develop the theory further. The subsequent formulas do not involve such a split. Moreover, the method is insensitive to the rank of the Lagrange anchor producing a well-defined path-integral quantization in the irregular case as well.

Example 1. *Let us illustrate the definitions above by an example of a Lagrangian gauge theory with action* $S(x)$*. The equations of motion read*

$$T \equiv dS(x) = 0\,, \tag{2.14}$$

so that the dynamics bundle $\mathcal{E}$ *is given by the cotangent bundle* T^*M *of the space of all histories. The canonical Lagrange structure, resulting in standard quantization, is given by the exterior differential* $d : \Gamma(\wedge^n T^*M) \to \Gamma(\wedge^{n+1}T^*M)$*. The defining condition for the Lagrange structure (2.12) takes the form*

$$dT = d^2 S \equiv 0. \tag{2.15}$$

The Lagrange anchor is defined by the identical homomorphism

$$V = \mathrm{id} : TM \to TM\,, \tag{2.16}$$

and hence the Lagrange structure is regular and complete. Suppose the action S *is gauge invariant. Then there exist a set of gauge algebra generators defining an* M*-bundle morphism* $R : \mathcal{F} \to TM$ *such that*

$$\langle R(\varepsilon), dS \rangle = 0 \tag{2.17}$$

for any gauge parameter $\varepsilon \in \Gamma(\mathcal{F})$*. So, equations (2.14) appear to be linearly dependent. Differentiating the last identity w.r.t. some connection* ∇ *on* $\mathcal{F}\otimes TM$*, we arrive at Rel. (2.9) with* $U^j_{\alpha i} = \nabla_i R^j_\alpha$.

Thus, we see that for ordinary Lagrangian gauge theories the dynamics bundle coincides with the cotangent bundle ($\mathcal{E} = T^*M$), the Noether identity bundle coincides with the gauge algebra bundle ($\mathcal{F} = \mathcal{G}$), and the generators of gauge symmetry coincide with the generators of Noether's identities ($R = Z$). For a general regular system of type $(1,1)$ neither of these coincidences should necessarily occur. For instance, it is possible to have gauge invariant, but linearly independent equations of motion; and conversely, a theory may have linearly dependent equations of motion without gauge symmetry.

2.4 S_∞-algebras

Recall that the conventional BV formalism for a Lagrangian gauge theory starts by introducing ghost fields to every gauge symmetry, and then an antifield for every field. The space of all fields and antifields is endowed with the canonical odd Poisson bracket $(\,\cdot\,,\cdot\,)$ and the original action functional is extended to the master action S defined as a proper solution to the classical master equation $(S, S) = 0$. The classical BRST differential $Q = (S, \cdot)$, being a nilpotent derivation of the odd Poisson algebra of functions, incorporates then both the dynamical equations and the gauge algebra structure. Thus, the odd Poisson geometry provide a natural framework for the BV field-antifield formalism.[5] In [7], we have shown that the quantization of general non-Lagrangian gauge theories call for a strongly homotopical version of the odd Poisson algebras.

Definition 4. *An S_∞-algebra (S for Schouten) is a $\mathbb{Z}_2$-graded, supercommutative, and associative algebra A endowed with a sequence of odd linear maps $S_n : A^{\otimes n} \to A$ such that*

(a) $S_n(\ldots, a_k, a_{k+1}, \ldots) = (-1)^{\epsilon(a_k)\epsilon(a_{k+1})} S_n(\ldots, a_{k+1}, a_k, \ldots)$,
$\epsilon(a)$ *being the parity of a homogeneous element* $a \in A$.

(b) $a \mapsto S_n(a_1, \ldots, a_{n-1}, a)$ *is a derivation of* A *of the parity* $1 + \sum_{k=1}^{n-1} \epsilon(a_k) \pmod 2$.

(c) For all $n \geq 0$,

$$\sum_{k+l=n} \sum_{(k,l)-\text{shufle}} (-1)^\epsilon S_{l+1}(S_k(a_{\sigma(1)}, \ldots, a_{\sigma(k)}), a_{\sigma(k+1)}, \ldots, a_{\sigma(k+l)}) = 0\,,$$

where $(-1)^\epsilon$ *is the natural sign prescribed by the sign rule for permutation of homogeneous elements* $a_1, \ldots, a_n \in A$.

Recall that a (k,l)-shuffle is a permutation of indices $1, 2, \ldots, k+l$ satisfying $\sigma(1) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(k+l)$.

When $S_0 = 0$ we speak about a *flat* S_∞-algebra. In that case $S_1 : A \to A$ is a differential with $(S_1)^2 = 0$, and S_2 induces an odd Poisson structure on the cohomology of

[5] In the physical literature the odd Poisson manifolds are usually called *anti-Poisson* manifolds. Correspondingly, the odd Poisson brackets are referred to as *antibrackets*. On the other hand, in mathematics the odd Poisson algebras (brackets) are also known under the names of Schouten or Gerstenhaber algebras (brackets).

S_1. An odd Poisson algebra can be regarded as an S_∞-algebra with bracket $S_2 : A\otimes A \to A$ and all other $S_k = 0$. In fact, properties (a) and (c) characterize L_∞-algebras. We refer to [12] for a recent discussion of S_∞-algebras.

It turns out that any Lagrange structure of type (m, n) gives rise to a flat S_∞-algebra on the supercommutative algebra of sections

$$A = \Gamma\big(\wedge^\bullet \mathcal{E} \otimes \bigotimes_{k=1}^{m} S^\bullet(\Pi^k \mathcal{E}_{-k}) \otimes \bigotimes_{l=1}^{n} S^\bullet(\Pi^{l+1}\mathcal{E}_l)\big)\,. \tag{2.18}$$

Here $S^\bullet$ stands for symmetric tensor powers (in the $\mathbb{Z}_2$-graded sense) and Π denotes the parity reversion operation, i.e., $\Pi\mathcal{E}$ is a vector bundle over M whose fibers are odd linear spaces. By definition, $\Pi^2 = \mathrm{id}$ and $S^\bullet(\Pi\mathcal{E}) = \wedge^\bullet\mathcal{E}$.

In the next section, applying the machinery of BRST theory, we give an explicit description for S_∞-algebras associated with $(1, 1)$-type Lagrange structures. Extension to the general Lagrange structures is straightforward.

3. BRST complex

3.1 An ambient symplectic supermanifold

Let $(\mathcal{E}, T, d_\mathcal{E})$ be a regular Lagrange structure corresponding to the four-term sequence (2.8). Following the general line of ideas of BRST theory, we realize M — the space of all histories — as the body of a graded supermanifold $\mathcal{N}$. The latter is chosen to be the total space of the following graded vector bundle over M:

$$\Pi(\mathcal{F} \oplus \mathcal{F}^*) \oplus T^*M \oplus \Pi(\mathcal{E} \oplus \mathcal{E}^*) \oplus (\mathcal{G} \oplus \mathcal{G}^*)\,. \tag{3.1}$$

Here $\mathcal{F}$, $\mathcal{E}$, and $\mathcal{G}$ are the bundles of gauge algebra, dynamical equations and the Noether identities, respectively. The base M is imbedded into (3.1) as the zero section. In addition to the Grassman parity the fibers of (3.1) are graded by *ghost number* valued in integers. To avoid cumbersome sign factors, we will assume the base M to be an ordinary (even) manifold that corresponds to the case of gauge systems without fermionic degrees of freedom. Then the Grassman parities of fibers correlate with their ghost numbers in a rather simple way: the even coordinates have even ghost numbers, while the odd coordinates have odd ghost numbers. The supermanifold $\mathcal{N}$ is also endowed with an $\mathbb{N}$-grading called the *momentum degree* (or m-degree for short).

It is convenient to arrange the information about all the aforementioned gradings of local coordinates in a single table 1.

Upon splitting all the coordinates into the "position coordinates" $\varphi^I = (x^i, c^\alpha, \eta_a, \xi_A)$ and "momenta" $\bar\varphi_J = (\bar x_i, \bar c_\alpha, \bar\eta^a, \bar\xi^A)$ the assignment of gradings becomes easy to see

$$\begin{aligned} &\mathrm{gh}(\bar\varphi_I) = -\mathrm{gh}(\varphi^I)\,, \qquad && \epsilon(\bar\varphi_I) = \epsilon(\varphi^I)\,, \\ &\mathrm{Deg}(\bar\varphi_I) = 1\,, && \mathrm{Deg}(\varphi^I) = 0\,. \end{aligned} \tag{3.2}$$

base and fibers	M	T^*M	$\mathcal{F}$	$\mathcal{F}^*$	$\mathcal{E}$	$\mathcal{E}^*$	$\mathcal{G}$	$\mathcal{G}^*$
local coordinates	x^i	$\bar{x}_j$	c^α	$\bar{c}_\beta$	η_a	$\bar{\eta}^b$	ξ_A	$\bar{\xi}^B$
ϵ = Grassman parity	0	0	1	1	1	1	0	0
gh = ghost number	0	0	1	-1	-1	1	-2	2
Deg = momentum degree	0	1	0	1	0	1	0	1

Table 1: The gradings of local coordinates on $\mathcal{N}$.

Let us denote by $C^\infty(\mathcal{N})$ the supercommutative algebra of "smooth functions" on $\mathcal{N}$. By definition, the generic element of $C^\infty(\mathcal{N})$ is given by a formal power series in the fiber coordinates with coefficients in $C^\infty(M)$.

Fixing a linear connection $\nabla = \nabla_{\mathcal{F}} \oplus \nabla_{\mathcal{E}} \oplus \nabla_{\mathcal{G}}$ on $\mathcal{F} \oplus \mathcal{E} \oplus \mathcal{G}$, we endow $\mathcal{N}$ with the exact symplectic structure

$$\omega = d(\bar{x}_i dx^i + \bar{c}_\alpha \nabla c^\alpha + \bar{\eta}^a \nabla \eta_a + \bar{\xi}^A \nabla \xi_A)\,, \tag{3.3}$$

where

$$\nabla c^\alpha = dc^\alpha + dx^i \Gamma^\alpha_{i\beta} c^\beta\,, \tag{3.4}$$

and similar expressions are assumed for covariant differentials of η's and ξ's. Thus, $C^\infty(\mathcal{N})$ becomes a Poisson algebra; the nonvanishing Poisson brackets of local coordinates are given by

$$\begin{aligned}
&\{\bar{\eta}^b, \eta_a\} = \delta^b_a\,, \quad \{\bar{x}_i, \eta_a\} = \Gamma^b_{ia}\eta_b\,, \quad \{\bar{x}_i, \bar{\eta}^b\} = -\Gamma^b_{ia}\bar{\eta}^a\,,\\
&\{\bar{c}_\alpha, c^\beta\} = \delta^\beta_\alpha\,, \quad \{\bar{x}_i, c^\alpha\} = \Gamma^\alpha_{i\beta}c^\beta\,, \quad \{\bar{x}_i, \bar{c}_\beta\} = -\Gamma^\alpha_{i\beta}\bar{c}_\alpha\,,\\
&\{\bar{\xi}^A, \xi_B\} = \delta^A_B\,, \quad \{\bar{x}_i, \xi_A\} = \Gamma^B_{iA}\xi_B\,, \quad \{\bar{x}_i, \bar{\xi}^A\} = -\Gamma^A_{iB}\bar{\xi}^B\,,\\
&\{\bar{x}_i, x^j\} = \delta^j_i\,, \quad \{\bar{x}_i, \bar{x}_j\} = R^b_{ija}\bar{\eta}^a\eta_b + R^\beta_{ij\alpha}c^\alpha\bar{c}_\beta + R^B_{ijA}\bar{\xi}^A\xi_B\,.
\end{aligned} \tag{3.5}$$

Here the structure functions determining the Poisson brackets of $\bar{x}_i$ and $\bar{x}_j$ are just the components of the curvature tensor of ∇.

Notice that the equations $\bar{\varphi}_I = 0$ define the Lagrangian submanifold

$$\mathcal{L} = \Pi(\mathcal{F} \oplus \mathcal{E}) \oplus \mathcal{G} \subset \mathcal{N}\,, \tag{3.6}$$

and the supercommutative algebra of functions $C^\infty(\mathcal{L})$ is naturally isomorphic to the algebra (2.18) with $m = n = 1$.

3.2 BRST charge

It turns out that all the ingredients of a classical gauge system as well as a Lagrange structure can be naturally incorporated into a single object Ω, the classical BRST charge.[6] By definition [7], the BRST charge Ω is an element of the Poisson algebra $C^\infty(\mathcal{N})$ such that

(i) $\epsilon(\Omega) = 1\,,\quad \mathrm{gh}(\Omega) = 1,\quad \mathrm{Deg}(\Omega) > 0;$

[6] Relevance of this terminology is explained in the next subsection.

(ii) $\Omega = \bar{\eta}^a T_a + c^\alpha R^i_\alpha \bar{x}_i + \bar{\xi}^A Z^a_A \eta_a + \bar{\eta}^a V^i_a \bar{x}_i + \cdots$;

(iii) $\{\Omega, \Omega\} = 0$.

The dots in (ii) refer to the terms which are at least linear in η_a and $\bar{c}_\alpha$ or at least quadratic in $\bar{x}_i$. Equation (iii) is known as the (classical) master equation. Conditions (i)-(iii) determine Ω up to a canonical transformation of $C^\infty(\mathcal{N})$. The existence of Ω is proved by standard tools of homological perturbation theory [7].

It is instructive to consider expansion of Ω in powers of momenta. In view of (i) the expansion starts with terms linear in $\bar{\varphi}$, i.e.,

$$\Omega = \sum_{k=1}^{\infty} \Omega_k \,, \qquad \mathrm{Deg}(\Omega_k) = k \,. \tag{3.7}$$

On substituting (3.7) into the master equation (iii), we get

$$\{\Omega_1, \Omega_1\} = 0 \,, \quad \{\Omega_1, \Omega_2\} = 0 \,, \quad \{\Omega_2, \Omega_2\} = -2\{\Omega_2, \Omega_3\} \,, \quad \text{etc} \,. \tag{3.8}$$

We see that the leading term $\Omega_1 = \Omega^I \bar{\varphi}_I$ gives rise to the *homological vector field* on $\mathcal{L}$,

$$Q \equiv \Omega^I \frac{\partial}{\partial \varphi^I} = T_a \frac{\partial}{\partial \eta_a} + c^\alpha R^i_\alpha \frac{\partial}{\partial x^i} + \eta_a Z^a_A \frac{\partial}{\partial \xi_A} + \cdots \,, \tag{3.9}$$

which carries all the information about the classical system itself, with no regard to the Lagrange structure.[7] Evaluating the nilpotency condition $Q^2 = 0$ to lowest order in fiber coordinates, one immediately recovers Rels. (2.9), (2.10) characterizing $T = 0$ as a set of gauge invariant and linearly dependent equations of motion, with R and Z being the generators of gauge transformations and Noether identities, respectively.

The Lagrange anchor $V : \mathcal{E}^* \to TM$ defining the Lagrange structure for the classical system (3.9) enters the next term

$$\Omega_2 = \Omega^{IJ}(\varphi)\bar{\varphi}_I \bar{\varphi}_J = \bar{\eta}^a V^i_a \bar{x}_i + \cdots \,. \tag{3.10}$$

Relations (3.8) characterize Ω_2 as a weak anti-Poisson structure on $\mathcal{L}$, i.e., Q-invariant, odd bivector field satisfying the Jacobi identity up to homotopy. The corresponding "weak" antibracket reads

$$(a, b) \equiv \{\{\Omega_2, a\}, b\} \,, \qquad a, b \in C^\infty(\mathcal{L}) \,. \tag{3.11}$$

Examining the Jacobi identity for this bracket, one finds

$$\begin{gathered}(a,(b,c)) + (-1)^{\epsilon(b)\epsilon(c)}((a,c),b) + (-1)^{\epsilon(a)(\epsilon(b)+\epsilon(c))}((b,c),a) = \\ -S_3(Qa,b,c) - (-1)^{\epsilon(a)\epsilon(b)} S_3(a,Qb,c) - (-1)^{(\epsilon(a)+\epsilon(b))\epsilon(c)} S_3(a,b,Qc) \\ -QS_3(a,b,c) \,,\end{gathered} \tag{3.12}$$

where we have introduced the following notation:

$$S_n(a_1, a_2, \ldots, a_n) \equiv \{\ldots\{\{\Omega_n, a_1\}a_2\}, \ldots, a_n\} \,, \qquad a_k \in C^\infty(\mathcal{L}) \,. \tag{3.13}$$

[7] In the usual BV theory the operator Q is known as the classical BRST differential [6, §8.5].

Evidently, the weak antibracket (3.11) induces a genuine antibracket in the Q-cohomology.

It is Rel. (3.13) that defines the aforementioned S_∞-structure on the supercommutative algebra $C^\infty(\mathcal{L})$: By definition, each S_n is a symmetric multi-differentiation of $C^\infty(\mathcal{L})$ and the generalized Jacobi identities for the collection of maps $\{S_n\}$ readily follow from the master equation $\{\Omega, \Omega\} = 0$ for the BRST charge. Since $\mathrm{Deg}\,(\Omega) > 0$, this S_∞-algebra is flat.

3.3 Hamiltonian interpretation

In the conventional BFV approach, the BRST charge arises as a tool for quantizing first-class constrained Hamiltonian systems. A glance at table 1 is enough to see that the spectrum of ghost numbers corresponds to that of the BFV-BRST formalism for a first-class constrained Hamiltonian system with linearly dependent constraints [6]. In order to make this interpretation more explicit, let us combine the local coordinates with ghost numbers 1 and -1 into the ghost coordinates $\mathcal{C}^I = (\bar{\eta}^a, c^\alpha)$ and ghost momenta $\bar{\mathcal{P}}_I = (\eta_a, \bar{c}_\alpha)$, respectively. In this notation the above BRST charge (3.7) can be rewritten as

$$\Omega = \mathcal{C}^I \Theta_I(x, \bar{x}) + \bar{\mathcal{P}}_I \Xi^I_A(x, \bar{x})\xi^A + \frac{1}{2}\bar{\mathcal{P}}_K U^K_{IJ}(x, \bar{x})\mathcal{C}^J\mathcal{C}^I + o(\bar{\mathcal{P}}^2, \xi^2)\,, \tag{3.14}$$

where the expansion coefficients $\Theta_I = (\widetilde{T}_a, \widetilde{R}_\alpha)$ and $\Xi^I_A = (\widetilde{Z}^a_A, Z^\alpha_A)$, playing the role of first-class constraints and their null-vectors, are given by the formal power series in $\bar{x}$'s:

$$\begin{aligned} \widetilde{T}_a(x, \bar{x}) &= T_a(x) + V^i_a(x)\bar{x}_i + o(\bar{x}^2)\,, \\ \widetilde{R}_\alpha(x, \bar{x}) &= R^i_\alpha(x)\bar{x}_i + o(\bar{x}^2)\,, \\ \widetilde{Z}^a_A(x, \bar{x}) &= Z^a_A(x) + o(\bar{x})\,. \end{aligned} \tag{3.15}$$

To lowest order in $\mathcal{C}$'s, eqs. (3.8) reproduce the standard involution relations for a set of reducible first-class constraints w.r.t. the canonical Poisson bracket on T^*M:

$$\{\Theta_I, \Theta_J\} = U^K_{IJ}\Theta_K\,, \qquad \Xi^I_A\Theta_I = 0\,. \tag{3.16}$$

From the regularity condition it follows immediately that the number of independent first-class constraints $\Theta_I \approx 0$ is equal to $\dim M$. In physical terms, one can interpret this fact concluding that the Hamiltonian system under consideration has no physical degrees of freedom. From the geometrical viewpoint, this implies that the equations $\Theta_I = 0$ define a Lagrangian submanifold $L \subset T^*M$; more accurately, L is a formal Lagrangian submanifold as we are not concerned with convergence of the formal series (3.15).

One can also regard the constraints $\Theta_I \approx 0$ as a formal deformation of those given by the leading terms of expansions (3.15) in the "direction" of the Lagrange anchor V. From this standpoint, the Lagrange structure is just the infinitesimal of deformation of the Lagrangian submanifold $L_0 \subset T^*M$ defined by the "bare" first-class constraints $T_a(x) \approx 0$ and $R^i_\alpha(x)\bar{x}_i \approx 0$.

Associated with the first-class constraints $\Theta_I \approx 0$ is the Hamiltonian action on the cotangent bundle of the space of all histories

$$S[\lambda, x, \bar{x}] = \int_{t_1}^{t_2} dt(\bar{x}_i\dot{x}^i - \lambda^I\Theta_I(x, \bar{x}))\,. \tag{3.17}$$

The action describes a pure topological field theory having no physical evolution w.r.t. to t. It should be emphasized, that the "time" t is an auxiliary $(d+1)$-st dimension, which has nothing to do with the evolution parameter in the (differential) equations of motion $T_a = 0$. The true physical time is among the original d dimensions.

The model (3.17) is invariant under the standard gauge transformations generated by the first-class constraints and their null-vectors (3.15):

$$\delta_\varepsilon x^i = \{x^i, \Theta_I\}\varepsilon^I , \qquad \delta_\varepsilon \bar{x}_i = \{\bar{x}_i, \Theta_I\}\varepsilon^I , \\ \delta_\varepsilon \lambda^I = \dot{\varepsilon}^I - \lambda^K U^I_{KJ}\varepsilon^J + \Xi^I_A \varepsilon^A . \tag{3.18}$$

Here $\varepsilon^I = (\varepsilon^a, \varepsilon^\alpha)$ and ε^A are infinitesimal gauge parameters, and the structure functions $U^I_{KJ}(\phi)$ are defined by (3.16).

Imposing the zero boundary conditions on the momenta

$$\bar{x}_i(t_1) = \bar{x}_i(t_2) = 0 , \tag{3.19}$$

one can see [8] that the classical dynamics of the model (3.17) are equivalent to those described by the original (non-)Lagrangian equations $T_a = 0$.

Example 2. *Given the Lagrangian equations of motion (2.14), Lagrange anchor (2.16), and gauge symmetry generators (2.17), we have the following set of first-class constraints on the phase space of fields and sources:*

$$\widetilde{T}_i = \partial_i S + \bar{x}_i , \qquad \widetilde{R}_\alpha = R^i_\alpha \bar{x}_i . \tag{3.20}$$

From the definition of gauge algebra it readily follows that

$$\{\widetilde{T}_i, \widetilde{T}_j\} = 0, \quad \{\widetilde{R}_\alpha, \widetilde{R}_\beta\} = U^\gamma_{\alpha\beta}\widetilde{R}_\gamma + U^i_{\alpha\beta}\widetilde{T}_i, \quad \{\widetilde{R}_\alpha, \widetilde{T}_i\} = U^j_{\alpha i}\widetilde{T}_j, \tag{3.21}$$

where $U^j_{\alpha i} = \partial_i R^j_\alpha$ and $U^i_{\alpha\beta} = \bar{x}_j W^{ij}_{\alpha\beta}(x)$. Evidently, the constraints (3.20) are reducible,

$$\widetilde{R}_\alpha = R^i_\alpha \widetilde{T}_i , \tag{3.22}$$

and we can take $\{\widetilde{T}_i\}$ as a complete set of independent first-class constraints. The corresponding Hamiltonian action (3.17) on the phase space of fields and sources reads

$$S_H[x, \bar{x}, \lambda] = \int_{t_1}^{t_2} dt \left(\bar{x}_i \dot{x}^i - \lambda^i (\partial_i S(x) + \bar{x}_i) \right) . \tag{3.23}$$

Excluding the momenta from this action by means of equations of motion $\delta S/\delta\lambda^i = 0$, we obtain

$$S_H[x] = \int_{t_1}^{t_2} dt\, \dot{x}^i \partial_i S(x) = S(x(t_2)) - S(x(t_1)) . \tag{3.24}$$

The latter action describes two copies of the original Lagrangian theory corresponding to the ends of the "time" interval $[t_1, t_2]$. As there is no coupling between the fields $x(t_1)$ and $x(t_2)$, one can consistently restrict dynamics to either subsystem with action $\pm S[x]$. This proves classical equivalence of the topological theory with action (3.23) to (the two copies of) the Lagrangian theory with action $S[x]$.

3.4 Physical observables

The Poisson action of Ω on $\mathcal{N}$ makes the space $C^\infty(\mathcal{N})$ into a cochain complex graded by ghost number: A function A is said to be BRST-closed if $\{\Omega, A\} = 0$ and BRST-exact if $A = \{\Omega, B\}$ for some B. Let $H^n(\Omega)$ denote the corresponding cohomology groups. As usual, the space of *physical observables* is identified with the group $H^0(\Omega)$, BRST cohomology at ghost number zero.

It can be shown [7] that the cohomology class of any BRST cocycle A with ghost number zero is completely determined by its restriction to M, i.e., by the function $\bar{A} = A|_M$, and a function $O \in C^\infty(M)$ is the restriction of some BRST cocycle iff

$$\langle R(\varepsilon), dO\rangle|_\Sigma = 0\,, \qquad \forall \varepsilon \in \Gamma(\mathcal{F})\,. \tag{3.25}$$

The trivial BRST cocycles are precisely those for which $O|_\Sigma = 0$. Thus, to any on-shell gauge-invariant function $O \in C^\infty(M)$ one can associate a BRST cocycle and vice versa. Let $[A] \in H^0(\Omega)$ and $x_0 \in \Sigma$, then the map

$$[A] \quad \mapsto \quad \langle A\rangle \equiv \bar{A}(x_0) \in \mathbb{R} \tag{3.26}$$

establishes the isomorphism $H^0(\Omega) \simeq \mathbb{R}$. Since $\Sigma \subset M$ is a connected submanifold and the distribution R acts on Σ transitively (see Corollary 1), the map (3.26) does not depend on the choice of $x_0 \in \Sigma$. By definition, $\langle A\rangle$ is the classical expectation value of the physical observable A.

4. Quantization

In previous sections, we have described the procedure that assigns a BRST complex to any dynamical system, be it Lagrangian or not. The input data needed for constructing such a complex are the classical equations of motion and the Lagrange structure. This BRST complex has a clear physical interpretation as that resulting from the BFV-BRST quantization of the topological sigma-model (3.17), whose target space is the cotangent bundle of the space of all histories. By construction, the classical dynamics of this effective topological theory are equivalent to the original ones for any choice of the Lagrange structure. Quantizing now the model (3.17) by the usual BFV-BRST method, we induce some quantization of the original (non-)Lagrangian theory; in so doing, different Lagrange structures may result in different quantizations of one and the same classical model.

Below, we start applying the standard prescriptions of the BFV-BRST operator quantization to the constrained Hamiltonian system (3.17). What remains to specify is a convenient representation. Here we prefer to work in the coordinate (Schrödinger) representation, whereby a quantum state is described by a wave-function on the ghost-extended space of all histories. Then a physical wave-function is nothing but the probability amplitude to find a system developing according to a given history. For the Lagrangian systems, this amplitude is simply given by the exponential of the action functional multiplied by $i/\hbar$. In the non-Lagrangian case, however, it may be a more general distribution, whose form strongly depends on the choice of a Lagrange anchor. (see examples in section 6).

A consistent consideration of physical states in the coordinate representation is known to require further enlargement of the extended phase space by the so-called *nonminimal variables* [6]. These do not actually change the physical content of the theory as one gauges them out by adding appropriate terms to the original BRST charge. The nonminimal sector just serves to bring the physical states to the ghost-number zero subspace where one can endow them with a well-defined inner product. We will not dwell on that in details, referring to the textbook [6]. From now on, Ω will stand for the total (i.e., nonminimal) BRST charge and the phase space $\mathcal{N}$ will include both minimal and nonminimal variables.

4.1 Quantum BRST cohomology

Upon canonical quantization each function on $\mathcal{N}$ turns to a linear operator acting in a complex Hilbert space $\mathcal{H}$:

$$C^\infty(\mathcal{N}) \ni F \quad \mapsto \quad \hat{F} \in \mathrm{End}(\mathcal{H})\,. \tag{4.1}$$

A crucial step in the operator BFV-BRST quantization [6] is assigning a nilpotent operator $\hat{\Omega}$ to the classical BRST charge (3.14). The quantum symbol of the BRST operator $\hat{\Omega}$ is supposed to have the form

$$\Omega(\varphi,\bar{\varphi},\hbar) = \sum_{k=0}^{\infty} \hbar^k \Omega^{(k)}(\varphi,\bar{\varphi})\,, \tag{4.2}$$

where the leading term $\Omega^{(0)}$ is given by (3.14) and the higher orders in $\hbar$ are determined from the requirements of hermiticity and nilpotency:

$$\hat{\Omega}^\dagger = \hat{\Omega}\,, \qquad \hat{\Omega}^2 = 0\,. \tag{4.3}$$

It may well happen that no $\hat{\Omega}$ exists satisfying these two conditions, in which case one speaks about *quantum anomalies*. In what follows we assume our theory to be anomaly free so that both equations (4.3) hold true.

In addition to the nilpotent BRST charge, the full BRST algebra involves also the anti-Hermitian ghost-number operator $\hat{\mathcal{G}}$ such that $[\hat{\mathcal{G}},\hat{F}] = \mathrm{gh}(F)\hat{F}$ for any homogeneous $\hat{F}$. In particular,

$$[\hat{\mathcal{G}},\hat{\Omega}] = \hat{\Omega}\,, \qquad \hat{\mathcal{G}}^\dagger = -\hat{\mathcal{G}}\,. \tag{4.4}$$

Given the BRST algebra (4.3), (4.4) one has two BRST complexes.

The first one is given by the space of quantum state $\mathcal{H}$ with $\hat{\Omega}$ playing the role of coboundary operator. Under certain assumptions [6] the space $\mathcal{H}$ splits as a sum $\mathcal{H} = \oplus_{n\in\mathbb{Z}}\mathcal{H}^n$ of eigenspaces of $\hat{\mathcal{G}}$ with definite real ghost number. Then $\hat{\Omega} : \mathcal{H}^n \to \mathcal{H}^{n+1}$ is the cochain complex of quantum states graded by ghost number. By $H^n_{\mathrm{st}}(\Omega)$ we denote the n-th group of the BRST-state cohomology.

Associated with the BRST complex of quantum states is the complex of quantum operators $\mathrm{End}(\mathcal{H}) = \oplus_{n\in\mathbb{Z}}\mathrm{End}^n(\mathcal{H})$. By definition, $\hat{F} \in \mathrm{End}^n(\mathcal{H})$, iff $\mathrm{ad}_{\hat{\mathcal{G}}}\hat{F} \equiv [\hat{\mathcal{G}},\hat{F}] = n\hat{F}$. The corresponding coboundary operator $\mathrm{ad}_{\hat{\Omega}} : \mathrm{End}^n(\mathcal{H}) \to \mathrm{End}^{n+1}(\mathcal{H})$ acts by the rule $\mathrm{ad}_{\hat{\Omega}}\hat{F} = [\hat{\Omega},\hat{F}]$. The n-th group of the BRST-operator cohomology is denoted by $H^n_{\mathrm{op}}(\Omega)$.

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The algebra of *quantum physical observables* and the space of *quantum physical states* are then identified with the corresponding BRST-cohomology at ghost number zero and the physical dynamics are described in terms of the $H^0_{\rm op}(\Omega)$-module $H^0_{\rm st}(\Omega)$.

Since the BRST charge is Hermitian, the inner product on $\mathcal{H}$ induces that on the space of physical states $H^0_{\rm op}(\Omega)$. In many interesting cases, however, the induced inner product appears to be ill defined and needs regularization. The most popular recipe to get a regular inner product is to fold the BRST-closed operator $e^{\frac{i}{\hbar}[\hat{\Omega},\hat{K}]}\sim 1$ between a pair of BRST-closed states $|\Psi_1\rangle, |\Psi_2\rangle \in \mathcal{H}$:

$$\langle\Psi_1|\Psi_2\rangle_K = \langle\Psi_1|e^{\frac{i}{\hbar}[\hat{\Omega},\hat{K}]}|\Psi_2\rangle\,, \tag{4.5}$$

K being an appropriate gauge-fixing fermion of ghost number -1. Evidently, the last expression passes to the BRST cohomology and is independent of a particular choice of K. (More precisely, it depends only on the homotopy class of K in the variety of all gauge-fixing fermions providing finiteness of (4.5).) Now the *quantum average* of a physical observable $[\hat{\mathcal{O}}]\in H^0_{\rm op}(\Omega)$ relative to a physical $[|\Psi\rangle]\in H^0_{\rm st}(\Omega)$ is given by

$$\langle\mathcal{O}\rangle = \frac{\langle\Psi|\hat{\mathcal{O}}|\Psi\rangle_K}{\langle\Psi|\Psi\rangle_K}\,. \tag{4.6}$$

4.2 Generalized Schwinger-Dyson equations

As the Hamiltonian theory we deal with is topological, it might be naively expected that $\dim_{\mathbb{C}} H^0_{\rm st}(\Omega) = 1$, so that the space of physical states is spanned by a unique (up to equivalence) BRST-closed state $|\Phi\rangle\in\mathcal{H}$. This would be quite natural because the probability amplitude must be a unique distribution on the space of all histories with prescribed boundary conditions. Actually, it is not always the case in the BRST theory: The physical dynamics may have several copies in the BRST-cohomology (the $H^0_{\rm op}(\Omega)$-module $H^0_{\rm st}(\Omega)$ is generally reducible), and choosing one of them amounts to imposing extra conditions on the physical states [6, §14.2.6]. A guiding principle here is to provide a positive-definiteness of the inner product (4.5) on a superselected physical space.

To be more specific, consider quantization of the gauge system (3.5), (3.14) in the case where $\mathcal{N}$ is a superdomain endowed with canonical Poisson brackets, that is, in formulas (3.5), we just take ∇ to be a flat connection. Furthermore, we assume that the $\varphi\bar{\varphi}$-symbol of the quantum BRST charge (4.2) satisfies condition

$$\Omega(\varphi, 0, \hbar) = 0\,. \tag{4.7}$$

This property takes place for the leading (classical) term $\Omega_0 = \Omega(\varphi,\bar{\varphi},0)$ and we require that it holds true with account of all quantum corrections. Then, the state $|\Phi\rangle$ that is annihilated by all the momenta,

$$\hat{\bar{\varphi}}_I|\Phi\rangle = 0\,, \tag{4.8}$$

is annihilated by the BRST charge as well. After an appropriate polarization of the non-minimal sector [8], the state $|\Phi\rangle$ caries zero ghost number, and hence, defines a physical state. In the coordinate representation, for instance, we have $\Phi(\varphi) = c\in\mathbb{C}$. At first glance

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the amplitude Φ, being just a constant, has nothing to do with the original dynamics, but that is illusion: The state $|\Phi\rangle$ has an ill-defined norm in $\mathcal{H}$, so in order to calculate the quantum average of a physical observable, say $1 \in H^0_{\rm op}(\Omega)$, one has use the regularized inner product (4.5), but that brings an inevitable dependence of the BRST charge. In other words, the information about the original gauge system enters to the state $|\Phi\rangle$ implicitly, through passage to the BRST cohomology. To make this dependence more explicit one should consider the BRST-dual of the state $|\Phi\rangle$ (see [6, §14.5.5] for general definitions). The dual state looks like[8]

$$|\Phi'\rangle = |\psi\rangle \otimes |ghosts\rangle \tag{4.9}$$

and is *not* in general equivalent to the state $|\Phi\rangle$. In the coordinate representation the first factor $|\psi\rangle$, called the *matter state*, is described by a wave-function on M, while the second factor is given by a wave-function of all other coordinates. By definition, the matter state is annihilated by the quantum constraints:

$$\hat{\Theta}_I|\psi\rangle = 0\,, \tag{4.10}$$

where the $x\bar{x}$-symbols of the constraint operators $\hat{\Theta}_I$ are given by

$$\Theta_I(x,\bar{x},\hbar) = \frac{\partial\Omega(\varphi,\bar{\varphi},\hbar)}{\partial\mathcal{C}^I}\bigg|_{ghosts=0} \tag{4.11}$$

It is the state $|\psi\rangle$ that appears as physical state in Dirac's quantization method. The consistency of equations (4.10) implies the following commutation relations for the quantum constraints:

$$[\hat{\Theta}_I,\hat{\Theta}_J] = \hat{U}^K_{IJ}\hat{\Theta}_K \tag{4.12}$$

with $\hat{\Theta}$'s to the right of $\hat{U}$'s. In view of the last relation we can regard (4.10) as a non-abelian generalization of the Schwinger-Dyson equation to the case of non-Lagrangian gauge theories. The next example justifies this interpretation.

Example 3. *Upon canonical quantization in the coordinate representation the independent first-class constraints $\widetilde{T}_i$ in (3.20) turn to the pairwise commuting differential operators:*

$$\hat{\widetilde{T}}_i = \partial_i S(x) - i\hbar\frac{\partial}{\partial x^i}\,, \qquad [\hat{\widetilde{T}}_i,\hat{\widetilde{T}}_j] = 0\,. \tag{4.13}$$

Imposing these operators on the physical wave-function $\psi(x)$, we arrive at the well-known Schwinger-Dyson equation in coordinate representation

$$\left[\partial_i S(x) - i\hbar\frac{\partial}{\partial x^i}\right]\psi(x) = 0\,. \tag{4.14}$$

A unique (up to an overall constant) solution to this equation is given by the Feynman probability amplitude on M,

$$\psi(x) = e^{-\frac{i}{\hbar}S(x)}\,. \tag{4.15}$$

[8]Hereafter the term "ghosts" refers to all fields with nonzero ghost number.

One can also quantize the constraints (3.20) in the momentum representation, which is related to the coordinate one by the (functional) Fourier transform. The corresponding wave-function

$$Z(\bar{x}) = \int Dx\, \psi(x) e^{-\frac{i}{\hbar}x^i\bar{x}_i} = \int Dx\, e^{-\frac{i}{\hbar}(S(x)+x^i\bar{x}_i)} \tag{4.16}$$

is nothing but the generating functional of Green's functions with $\bar{x}$'s playing the role of classical sources.

In principle, one can use any copy of a single physical state in the BRST-state cohomology to compute the quantum average of a physical observable $[\mathcal{O}] \in H^0_{\mathrm{op}}(\Omega)$ by formula (4.6). It is also possible and is particularly convenient to use the asymmetric definition for the quantum averages:

$$\langle \mathcal{O} \rangle = \frac{\langle \Phi' | \hat{\mathcal{O}} | \Phi \rangle_K}{\langle \Phi' | \Phi \rangle_K} \tag{4.17}$$

In [8], it was shown that all such definitions give one and the same value $\langle \mathcal{O} \rangle$. Similarly to the BRST-state cohomology, the BRST-operator cohomology is essentially one-dimensional that allows one to establish a one-to-one correspondence between the physical states and physical observables. Namely, given a physical observable $\hat{\mathcal{O}}$, we define the physical state

$$|O\rangle = \hat{\mathcal{O}} |\Phi\rangle \,. \tag{4.18}$$

The latter is necessarily of the form $|O\rangle = \langle \mathcal{O} \rangle |\Phi\rangle + \hat{\Omega}|\Lambda\rangle$. Using the coordinate representation, we can rewrite (4.17) as

$$\langle \mathcal{O} \rangle = \frac{\langle \Phi' | O \rangle_K}{\langle \Phi' | 1 \rangle_K} = (\mathrm{const}) \int D\varphi O(\varphi) \Phi'_K(\varphi) \,. \tag{4.19}$$

The last expression enables us to treat the gauge-fixed probability amplitude $\Phi'_K(\varphi) = \langle \Phi' | \varphi \rangle_K$ as a linear functional on the space of physical observables represented by the physical states $O(\varphi) = \langle \varphi | O \rangle$.

4.3 Path-integral representation

Regarding the regulator $e^{\frac{i}{\hbar}[\hat{\Omega},\hat{K}]}$ in (4.5) as the evolution operator corresponding to the BRST-trivial Hamiltonian $\hat{H} = [\hat{\Omega}, \hat{K}]$, we can immediately write down the path-integral representation for the quantum average (4.6):

$$\langle \mathcal{O} \rangle = \frac{\langle \Phi | \hat{\mathcal{O}} e^{\frac{i}{\hbar}[\hat{\Omega},\hat{K}]} | \Phi \rangle}{\langle \Phi | e^{\frac{i}{\hbar}[\hat{\Omega},\hat{K}]} | \Phi \rangle} = (\mathrm{const}) \int \mathcal{D}\varphi \mathcal{D}\bar{\varphi}\, O(\varphi(1)) \exp \frac{i}{\hbar} \int_0^1 dt (\bar{\varphi}_I \dot{\varphi}^I - \{\Omega, K\}) \,. \tag{4.20}$$

Here the normalization constant is chosen in such a way that $\langle 1 \rangle = 1$ and integration extends over all fields obeying

$$\bar{\varphi}_I(0) = \bar{\varphi}_I(1) = 0 \,. \tag{4.21}$$

These boundary conditions follow directly from definition (4.8) of the physical state $|\Phi\rangle$. Because of (4.21) only the $\bar{\varphi}$-independent part

$$O(\varphi) = \mathcal{O}(\varphi, \bar{\varphi})|_{\bar{\varphi}=0} = O(x) + (\textit{ghost terms}) \tag{4.22}$$

of the physical observable $\mathcal{O}$ contributes to the path integral (4.20). It is not hard to see that the function (4.22) obeys to (and can be determined from) the following equation:

$$QO = \{\Omega_1, O\} = 0\,, \tag{4.23}$$

Q being the classical BRST differential (3.9).

Note that (4.20) is nothing but the usual Feynman's path integral for the topological sigma-model with action

$$S[\varphi, \bar{\varphi}] = \int_0^1 dt(\bar{\varphi}_I \dot{\varphi}^I - \{\Omega, K\})\,. \tag{4.24}$$

This can be viewed as resulting from the BFV quantization of the constrained Hamiltonian theory (3.17). If $\dim X = d$, where X is the initial space-time manifold, then (4.24) defines a topological field theory on the $(d+1)$-dimensional manifold $\widetilde{X} = X \times I$ with boundary. Suppose X is an orientable manifold, then so is $\widetilde{X} = X \times I$ and each of the two orientations of $\widetilde{X}$ induces opposite orientations on the connected components of the boundary $\partial\widetilde{X} = X_0 \cup X_1$; here $X_0 \simeq X \simeq X_1$ and the subscripts 0 and 1 refer to the different orientations of X.

As the model (4.24) is purely topological, there are no physical dynamics in the bulk of $\widetilde{X}$. Put differently, all the physical degrees of freedom, if any, are supported at the boundary $\partial\widetilde{X} = X_0 \cup X_1$, where they evolve according to the classical equations of motion $T_a = 0$; in so doing, the dynamics on X_0 and X_1 are completely independent of each other. Thus, the action (4.24) describes two copies of the same filed-theoretical model, which defer only by orientation of the space-time manifold X (two parallel universes).

This classical consideration can be further promoted to the quantum-mechanical level. Consider the *projected kernel* associated with the matter state (4.10). It can be defined by the path integral [6]:

$$\psi(x_1)\bar{\psi}(x_0) = \int D\varphi D\bar{\varphi}\, e^{\frac{i}{\hbar}S[\varphi,\bar{\varphi}]}\,, \tag{4.25}$$

where the sum runs over trajectories $(\varphi^I(t), \bar{\varphi}_J(t))$ subject to appropriate boundary conditions at $t = 0, 1$. In particular, $x_1 = x(1)$, $x_0 = x(0)$; the boundary conditions for the other variables can be found in [8]. According to our definitions, the state ψ describes a gauge invariant probability amplitude for a field theory on X_0. Then $\bar{\psi}$ must play the same role for X_1.[9] Multiplying ψ by $\bar{\psi}$, we get the right probability amplitude for the field theory on $X_0 \cup X_1$, as there is no interaction between the fields on X_0 and X_1 (correlations through the bulk of $\widetilde{X}$ are completely suppressed by gauge invariance).

Example 4. *Let us compute the quantum average (4.20) for the topological model (3.23), where the action S is not gauge invariant. A good gauge-fixing condition in the bulk is the derivative gauge*

$$\ddot{x}^i = 0\,. \tag{4.26}$$

[9] In the Lagrangian field theory, for example, the probability amplitude has the form $\psi = e^{\frac{i}{\hbar}S}$, where the action functional is given by the integral $S = \int_X L$ of some top form L (a Lagrangian density). Changing an orientation of X yields $S \to -S$, hence $\psi \to \bar{\psi}$.

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As with any abelian gauge theory, the ghost fields are decoupled from the matter ones and can thus be integrated out explicitly. The result is given by

$$\langle \mathcal{O} \rangle = c \int DxD\bar{x}D\lambda\delta[\ddot{x}]O(x(1))e^{\frac{i}{\hbar}S_H[x,\bar{x},\lambda]} \tag{4.27}$$

$$= c \int Dx\delta[\ddot{x}]O(x(1)) \exp \frac{i}{\hbar} (S[x(1)] - S[x(0)]) \tag{4.28}$$

$$= c' \int Dx(1)O(x(1))e^{\frac{i}{\hbar}S[x(1)]} = \langle O|\psi\rangle\langle\psi|1\rangle \,, \tag{4.29}$$

where

$$\psi(x) = e^{\frac{i}{\hbar}S[x]} \,, \qquad c' = c\langle\psi|1\rangle = c \int Dx(0)e^{-\frac{i}{\hbar}S[x(0)]} \,. \tag{4.30}$$

So, up to a normalization constant, the integral (4.27) gives the usual quantum average of an observable O in the Lagrangian theory with action S. Notice that one can arrive at the same result by imposing the derivative gauge on the Lagrange multiplier $\dot{\lambda}^i = 0$.

5. Augmentation

In previous sections, we have formulated the quantization procedure for (non-)Lagrangian gauge theories, which starts with the classical equations of motion and Lagrange structure as input data and results in the generalized Schwinger-Dyson equation for the probability amplitude on the space of all histories M. We have also seen that the amplitude admits a simple path-integral representation in terms of a Lagrangian topological field theory in the space-time with one more dimension. In this section, we derive an alternative path-integral representation for the probability amplitude of a (non-)Lagrangian theory in terms of some Lagrangian model on the same space-time manifold, but augmented with extra fields. The configuration space of the augmented field theory is taken to be the total space of the vector bundle $\mathcal{E}^* \to M$, the dual to the dynamics bundle $\mathcal{E}$; in so doing, the original configuration space M is embedded in $\mathcal{E}^*$ as the zero section. The augmentation procedure extends the original (non-)Lagrangian dynamics from M to $\mathcal{E}^*$ in such a way that the entire system becomes Lagrangian. We show that, at classical level, the augmented theory is equivalent to the original one provided that special boundary conditions are fixed for the augmentation fields. At quantum level, integrating the Feynman probability amplitude on $\mathcal{E}^*$ over the augmentation fields yields the probability amplitude on M.

5.1 An augmented BRST complex.

Augmentation of the original dynamics on M implies a consistent extension to $\mathcal{E}^*$ of the original equations of motion, gauge symmetries, Noether identities, and the Lagrange structure. As a practical matter, it is convenient to make these extensions not at the level of the space of all histories, but augmenting the ambient symplectic manifold $\mathcal{N}$, which already involves all necessary ghost fields of the original theory. The overall result of these extensions turns out to be just a "duplication" of the ambient manifold. More precisely, the manifold $\mathcal{N}$, considered as the total space of the vector bundle (3.1), is replaced with

fibers	$\mathcal{E}^*$	$\mathcal{E}$	$\mathcal{G}^*$	$\mathcal{G}$	T^*M	TM	$\mathcal{F}^*$	$\mathcal{F}$
fiber coordinates	y^a	$\bar{y}_b$	c^A	$\bar{c}_B$	η_i	$\bar{\eta}^j$	ξ_α	$\bar{\xi}^\beta$
ϵ =Grassman parity	0	0	1	1	1	1	0	0
gh = ghost number	0	0	1	-1	-1	1	-2	2
Deg = momentum degree	0	1	0	1	0	1	0	1
deg = resolution degree	1	0	1	0	1	0	1	0

Table 2: The gradings of augmentation fields.

$\mathcal{N}_{\text{aug}} = \mathcal{N} \oplus \Pi(\mathcal{N} \oplus TM)$. Table 2 contains the data on various gradings assigned to the fiber coordinates of $\Pi(\mathcal{N} \oplus TM)$:

In order to compare the ghost numbers of the new and old fields it is convenient to assemble the augmentation fields into "position coordinates" and "momenta":

$$\varphi_I^{\text{aug}} = (\eta_i, \xi_\alpha, y^a, c^A)\,, \qquad \bar{\varphi}^I_{\text{aug}} = (\bar{\eta}^i, \bar{\xi}^\alpha, \bar{y}_a, \bar{c}_A)\,. \tag{5.1}$$

Then we have

$$\begin{aligned} &\text{gh}(\varphi_I^{\text{aug}}) = -\text{gh}(\bar{\varphi}^I_{\text{aug}})\,, \qquad \epsilon(\varphi_I^{\text{aug}}) = \epsilon(\bar{\varphi}^I_{\text{aug}})\,, \\ &\text{Deg}(\bar{\varphi}^I_{\text{aug}}) = 1\,, \qquad \text{Deg}(\varphi_I^{\text{aug}}) = 0\,, \end{aligned} \tag{5.2}$$

and

$$\text{gh}(\varphi_I^{\text{aug}}) = \text{gh}(\bar{\varphi}_I) - 1\,, \qquad \text{gh}(\bar{\varphi}^I_{\text{aug}}) = \text{gh}(\varphi^I) + 1\,. \tag{5.3}$$

So, the "duplication" of the ambient manifold $\mathcal{N}$ is accompanied with reversion of parities and shift of ghost numbers.

As a next step, we extend the exact symplectic structure (3.3) on $\mathcal{N}$ to that on $\mathcal{N}_{\text{aug}}$ by setting

$$\omega_{\text{aug}} = \omega + d(\bar{\varphi}^I_{\text{aug}} \nabla \varphi_I^{\text{aug}})\,, \tag{5.4}$$

∇ being some connection on $\mathcal{N} \oplus TM$.

Finally, the original BRST charge Ω on $\mathcal{N}$ is extended to $\mathcal{N}_{\text{aug}}$ as

$$\Omega_{\text{aug}} = \Omega + \sum_{n=0}^{\infty} \Omega_n\,, \qquad \deg(\Omega_n) = n\,. \tag{5.5}$$

Here

$$\Omega_0 = \bar{\varphi}_I \bar{\varphi}^I_{\text{aug}} \tag{5.6}$$

and the higher orders in the resolution degree are determined from the master equation

$$\{\Omega_{\text{aug}}, \Omega_{\text{aug}}\} = 0\,. \tag{5.7}$$

Let us show that the last equation has a solution indeed. To this end, we introduce the following pair of nilpotent operators:

$$\begin{aligned} \delta &= \bar{\varphi}_I \frac{\partial}{\partial \varphi_I^{\text{aug}}}\,, \qquad \delta^2 = 0\,, \qquad \deg(\delta) = -1\,, \\ \delta^* &= \varphi_I^{\text{aug}} \frac{\partial}{\partial \bar{\varphi}_I}\,, \qquad (\delta^*)^2 = 0\,, \qquad \deg(\delta^*) = 1\,. \end{aligned} \tag{5.8}$$

It is straightforward to check that

$$\delta\delta^* + \delta^*\delta = N\,, \qquad N \equiv N_1 + N_2\,, \tag{5.9}$$

where the operator

$$N_1 = \bar{\varphi}_I \frac{\partial}{\partial \bar{\varphi}_I} \tag{5.10}$$

counts the momentum degree of the "old" variables (see table 1), while

$$N_2 = \varphi_I^{\mathrm{aug}} \frac{\partial}{\partial \varphi_I^{\mathrm{aug}}} \tag{5.11}$$

counts the resolution degree of augmentation fields. If we regard δ as the differential of the cochain complex $C^\infty(\mathcal{N}_{\mathrm{aug}})$, then δ^* becomes a homotopy for N with respect to δ. As a result, all the nontrivial δ-cocycles are nested in the subspace $\ker N \subset C^\infty(\mathcal{N}_{\mathrm{aug}})$.

Now applying the standard technique of homological perturbation theory [6], we can prove the following statement.

Proposition 1. *There is a unique BRST charge (5.5) satisfying the master equation (5.7) and the condition*

$$\delta^*(\Omega_{\mathrm{aug}} - \Omega - \Omega_0) = 0\,. \tag{5.12}$$

Proof. Expanding the master equation (5.7) with respect to the resolution degree, we arrive at the following sequence of equations:

$$\delta\Omega_{n+1} = B_n(\Omega_0, \dots, \Omega_n)\,, \qquad n \in \mathbb{N}\,, \tag{5.13}$$

where

$$B_n = P_n\Bigg(\{\Omega, \Omega_n\} + \sum_{s=0}^{n}\{\Omega_{n-s}, \Omega_s\}\Bigg)\,, \tag{5.14}$$

and P_n is the projector on the subspace of functions of resolution degree n. We can solve these equations in series starting with $\delta\Omega_1 = B_0$. Since $\deg B_n = n$ and the function $B_0 = \{\Omega, \Omega_0\}$ contains no terms of zero momentum degree w.r.t. the old variables, the operator N is invertible on the subspace $W \subset C^\infty(\mathcal{N}_{\mathrm{aug}})$ spanned by all B's. The condition $\delta B_n = 0$ is then necessary and sufficient for the n-th equation (5.13) to be solvable. The closedness of B_n is established by induction on n, just putting successive restrictions on the resolution degree of the Jacobi identity $\{\Omega, \{\Omega, \Omega\}\} \equiv 0$.

Finally, applying the operator δ^* to both sides of eq. (5.13) and using Rels. (5.9), (5.12), we get the following recurrent relations for the homogeneous components of Ω:

$$\Omega_{n+1} = \delta^*(N|_W)^{-1} B_n(\Omega_0, \dots, \Omega_n)\,. \tag{5.15}$$

Here we have used the fact that the operator δ^* commutes with N and, as a consequence, with $(N_W)^{-1}$. By construction, $\delta^*\Omega_n = 0$, $\forall n > 0$, so that the augmented BRST charge Ω_{aug} meets equation (5.12). □

In sequel we will need the following property of the augmented BRST charge.

Proposition 2. *If the augmented BRST charge (5.5) satisfies (5.12), then*

$$(\Omega_{\mathrm{aug}} - \Omega - \Omega_0)|_{\bar{\varphi}^I_{\mathrm{aug}}=0} = 0\,. \tag{5.16}$$

Proof. by induction on resolution degree.

5.2 Interpretation

Given the augmented BRST charge (5.5), one may ask what is a classical theory this BRST charge corresponds to (or can be derived from). According to the general definitions of section 3, the equations of motion, gauge symmetry and Noether identity generators, as well as the Lagrange anchor, enter the BRST charge as coefficients at lower powers of fiber coordinates with certain ghost numbers and momentum degrees (see relation (ii) at the beginning of section 3.2). So, to identify all the key ingredients of the underlying gauge dynamics we are going just to evaluate the appropriate terms in the augmented BRST charge. Thus, the equations of motion define the terms that are linear in fiber coordinates:

$$\mathbf{T}_a = \left.\frac{\partial\Omega_{\mathrm{aug}}}{\partial\bar{\eta}^a}\right|_{\mathcal{E}^*} = T_a(x) = 0\,, \qquad \mathbf{T}_i = \left.\frac{\partial\Omega_{\mathrm{aug}}}{\partial\bar{\eta}^i}\right|_{\mathcal{E}^*} = \nabla_i T_a y^a + o(y^2) = 0\,. \tag{5.17}$$

As is seen, the first group of equations coincides with the original equations of motion on M. The absence of y-contributions to these equations is guaranteed by Proposition 2. So, the original dynamics on M are completely decoupled from the augmented system (5.17). The second group of equations, being at least linear in y's, admit a trivial solution $y^a = 0$, which can be singled out by imposing zero boundary conditions on y's.

In general, the augmented equations of motion (5.17) are both gauge invariant and linearly dependent. It follows from definiens of section 3.2 that the gauge algebra generators are given by

$$\begin{aligned}
\mathbf{R}_\alpha &= \left.\frac{\partial^2\Omega_{\mathrm{aug}}}{\partial c^\alpha\partial\bar{x}_i}\right|_{\mathcal{E}^*}\frac{\partial}{\partial x^i} + \left.\frac{\partial^2\Omega_{\mathrm{aug}}}{\partial c^\alpha\partial\bar{y}_a}\right|_{\mathcal{E}^*}\frac{\partial}{\partial y^a} = R^i_\alpha(x)\frac{\partial}{\partial x^i} + o(y)\,,\\
\mathbf{R}_A &= \left.\frac{\partial^2\Omega_{\mathrm{aug}}}{\partial c^A\partial\bar{y}_a}\right|_{\mathcal{E}^*}\frac{\partial}{\partial y^a} + \left.\frac{\partial^2\Omega_{\mathrm{aug}}}{\partial c^A\partial\bar{x}_i}\right|_{\mathcal{E}^*}\frac{\partial}{\partial x^i} = Z^a_A(x)\frac{\partial}{\partial y^a} + o(y)\,,
\end{aligned} \tag{5.18}$$

so that

$$\begin{aligned}
\mathbf{R}_\alpha\mathbf{T}_a &= \mathbf{U}^b_{\alpha a}\mathbf{T}_b + \mathbf{U}^i_{\alpha a}\mathbf{T}_i\,, \qquad \mathbf{R}_A\mathbf{T}_a = \mathbf{U}^b_{Aa}\mathbf{T}_b + \mathbf{U}^i_{Aa}\mathbf{T}_i\,,\\
\mathbf{R}_\alpha\mathbf{T}_i &= \mathbf{U}^b_{\alpha i}\mathbf{T}_b + \mathbf{U}^j_{\alpha i}\mathbf{T}_j\,, \qquad \mathbf{R}_A\mathbf{T}_i = \mathbf{U}^b_{Ai}\mathbf{T}_b + \mathbf{U}^j_{Ai}\mathbf{T}_j\,,
\end{aligned} \tag{5.19}$$

for some structure functions $\mathbf{U}$. The Noether identities have the following form in the augmented theory:

$$\mathbf{Z}^a_A\mathbf{T}_a + \mathbf{Z}^i_A\mathbf{T}_i = 0\,, \qquad \mathbf{Z}^a_\alpha\mathbf{T}_a + \mathbf{Z}^i_\alpha\mathbf{T}_i = 0\,, \tag{5.20}$$

where

$$\begin{aligned}
\mathbf{Z}^a_A &= \left.\frac{\partial^2\Omega_{\mathrm{aug}}}{\partial\bar{\xi}^A\partial\eta_a}\right|_{\mathcal{E}^*} = Z^a_A(x) + o(y)\,, \qquad \mathbf{Z}^i_A = \left.\frac{\partial^2\Omega_{\mathrm{aug}}}{\partial\bar{\xi}^A\partial\eta_i}\right|_{\mathcal{E}^*} = o(y)\,,\\
\mathbf{Z}^i_\alpha &= \left.\frac{\partial^2\Omega_{\mathrm{aug}}}{\partial\bar{\xi}^\alpha\partial\eta_i}\right|_{\mathcal{E}^*} = R^i_\alpha(x) + o(y)\,, \qquad \mathbf{Z}^a_\alpha = \left.\frac{\partial^2\Omega_{\mathrm{aug}}}{\partial\bar{\xi}^\alpha\partial\eta_a}\right|_{\mathcal{E}^*} = o(y)\,.
\end{aligned} \tag{5.21}$$

As is seen from Rels. (5.18), there are two types of gauge symmetry transformations in the augmented theory. The first ones, generated by $\mathbf{R}_\alpha$, are just extensions to $\mathcal{E}^*$ of the original gauge symmetries. The second type transformations, generated by $\mathbf{R}_A$, start from the vertical vector fields on $\mathcal{E}^*$ associated with the Noether identity generators Z^a_A.

Looking at the generators of Noether's identities (5.21), one can observe a mirror inversion in the structure of the gauge symmetry generators: The generators R_α on M give rise to the Noether identity generators $\mathbf{Z}_\alpha$, while the generator $\mathbf{Z}_A$ is just a continuation of corresponding Noether identity generator from the original theory.

We thus conclude that the numbers of Noether's identities and gauge symmetries coincide in the augmented theory. Furthermore, the expansion in the augmentation fields y^i starts with the same terms for both sets of generators. And one can further deduce that the generators of identities (5.21) coincide with the generators of gauge symmetries (5.18). Such a pairing between Noether identities and gauge symmetries is characteristic for the Lagrangian dynamics.

To further elucidate the meaning of the augmented BRST charge in terms of the phase space of fields x^i, y^a and their sources $\bar{x}_i$, $\bar{y}_a$, we introduce the following collective notation:

$$\phi^{\bar{a}} = (x^i, y^a)\,, \qquad \bar{\phi}_{\bar{a}} = (\bar{x}_i, \bar{y}_a)\,, \qquad \bar{\eta}^{\bar{a}} = (\bar{\eta}^i, \bar{\eta}^a)\,. \tag{5.22}$$

Then the deformed phase-space constraints associated with the augmented equations of motion (5.17) are given by

$$\widetilde{\mathbf{T}}_{\bar{a}} = \left.\frac{\partial \Omega_{\mathrm{aug}}}{\partial \bar{\eta}^{\bar{a}}}\right|_{\mathcal{E}^*\oplus T^*M\oplus\mathcal{E}} = \mathbf{T}_{\bar{a}}(\phi) + V^{\bar{b}}_{\bar{a}}(\phi)\bar{\phi}_{\bar{b}} + \sum_{k=2}^{\infty} V^{\bar{b}_1\cdots\bar{b}_k}_{\bar{a}}(\phi)\bar{\phi}_{\bar{b}_1}\cdots\bar{\phi}_{\bar{b}_k} \approx 0\,. \tag{5.23}$$

According to our definitions, the coefficients $V^{\bar{b}}_{\bar{a}}(\phi)$ in (5.23) are to be identified with the components of the Lagrange anchor. Using the recurrent relations (5.15), we find

$$V = (V^{\bar{b}}_{\bar{a}}) = \begin{pmatrix} V^i_a(x) & \delta^b_a \\ \delta^i_j & 0 \end{pmatrix} + o(y)\,. \tag{5.24}$$

As is seen the augmented Lagrange anchor is always nondegenerate and its inverse has the form

$$\Lambda = V^{-1} = \begin{pmatrix} 0 & \delta^i_j \\ \delta^b_a & -V^i_a(x) \end{pmatrix} + o(y)\,. \tag{5.25}$$

To make contact with the definitions of section 2, we identify the total space of the tangent bundle $T\mathcal{E}^*$ with the total space of $\mathcal{E}^*\oplus TM\oplus\mathcal{E}^*$ and the total space of $T^*\mathcal{E}^*$ with that of $\mathcal{E}\oplus T^*M\oplus\mathcal{E}^*$ by making use the linear connection ∇ on $\mathcal{E}^*\to M$. Upon these identifications, the bundle map $T\mathcal{E}^*\to\mathcal{E}^*$ goes into the bundle map $\mathcal{E}^*\oplus TM\oplus\mathcal{E}^*\to\mathcal{E}^*$ (projection on the third factor) and the same is true for the cotangent bundle $T^*\mathcal{E}^*$. Now we can summarize the discussion above as follows.

Proposition 3. *The augmented BRST complex describes a complete Lagrange structure of type (1,1) associated to the on-shell exact sequence*

$$0\to\mathcal{V}\to T\mathcal{E}^*\to T^*\mathcal{E}^*\to\mathcal{V}\to 0\,, \tag{5.26}$$

where the gauge algebra (= Noether identity) bundle $\mathcal{V}$ is the vector bundle with the base $\mathcal{E}^*$, total space $\mathcal{F} \oplus \mathcal{E}^* \oplus \mathcal{G}$, and the bundle map $p : \mathcal{F} \oplus \mathcal{E}^* \oplus \mathcal{G} \to \mathcal{E}^*$ (projection on the second factor).

The completeness of the augmented Lagrange structure has two immediate consequences. First of all, the constraints (5.23) define a Lagrangian submanifold in the augmented phase space $\mathcal{E}^* \oplus T^*M \oplus \mathcal{E}$, so that the rest of the constraints, namely, the constraints

$$\begin{aligned} \widetilde{\mathbf{R}}_\alpha &= \left.\frac{\partial \Omega_{\mathrm{aug}}}{\partial c^\alpha}\right|_{\mathcal{E}^* \oplus T^*M \oplus \mathcal{E}} = \widetilde{R}_\alpha(x, \bar{x}) + o(y)\,, \\ \widetilde{\mathbf{R}}_A &= \left.\frac{\partial \Omega_{\mathrm{aug}}}{\partial c^A}\right|_{\mathcal{E}^* \oplus T^*M \oplus \mathcal{E}} = \widetilde{Z}^a_A(x, \bar{x})\bar{y}_a + o(y) \end{aligned} \tag{5.27}$$

associated with the gauge symmetry generators (5.18), (3.15), are given by linear combinations of (5.23). The second consequence is that, according to Theorem 1, the augmented equations of motion (5.17) are equivalent to Lagrangian ones.

To get an explicit expression for corresponding action functional, one has just to resolve the first-class constraints (5.23) with respect to momenta $\bar{\phi}_{\bar{a}}$. This can always be done at least perturbatively. As a starting point, we rewrite the constraint equations (5.23) in the following equivalent form:

$$\bar{\phi}_{\bar{a}} = -\Lambda^{\bar{b}}_{\bar{a}} \mathbf{T}_{\bar{b}} - \Lambda^{\bar{b}}_{\bar{a}} \sum_{k=2}^{\infty} V^{\bar{a}_1 \cdots \bar{a}_k}_{\bar{b}} \bar{\phi}_{\bar{a}_1} \cdots \bar{\phi}_{\bar{a}_k}\,, \tag{5.28}$$

where Λ is defined by relation (5.25). Then, taking $\bar{\phi} = -\Lambda \mathbf{T}$ as zero order approximation and iterating these equations ones and again, we finally arrive at the equivalent set of first-class constraints

$$\widetilde{\mathbf{T}}'_{\bar{a}} = \bar{\phi}_{\bar{a}} - \mathbf{T}'_{\bar{a}}(\phi) \approx 0\,, \tag{5.29}$$

where

$$\mathbf{T}'_{\bar{a}} = \sum_{k=1}^{\infty} F^{\bar{b}_1 \cdots \bar{b}_k}(\phi) \mathbf{T}_{\bar{b}_1}(\phi) \cdots \mathbf{T}_{\bar{b}_k}(\phi) = \widetilde{\Lambda}(\phi)^{\bar{b}}_{\bar{a}} \mathbf{T}_{\bar{b}}(\phi)\,. \tag{5.30}$$

The constraints (5.29), being resolved w.r.t. momenta $\bar{\phi}_{\bar{a}}$, are to be necessarily commuting,

$$\{\widetilde{\mathbf{T}}'_{\bar{a}}, \widetilde{\mathbf{T}}'_{\bar{b}}\} = 0\,, \tag{5.31}$$

that amounts to existence of an action functional $S(\phi)$ such that

$$\mathbf{T}'_{\bar{a}}(\phi) = \partial_{\bar{a}} S(\phi)\,. \tag{5.32}$$

Thus, the augmented equations of motion are equivalent to the Lagrangian equations (5.32) with $\widetilde{\Lambda}$ playing the role of integrating multiplier. Finally, using the standard homotopy operator for the exterior differential, we can reconstruct the action as

$$S(\phi) = \phi^{\bar{a}} \int_0^1 \mathbf{T}'_{\bar{a}}(s\phi) ds + (\mathrm{const})\,. \tag{5.33}$$

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Up to the second order in y's and an inessential additive constant the action reads

$$S(x,y) = T_a(x)y^a + G_{ab}(x)y^a y^b + o(y^3)\,. \tag{5.34}$$

Here the symmetric matrix

$$G_{ab} = V_a^i \nabla_i T_b + V_b^i \nabla_i T_a \tag{5.35}$$

can be thought of as a generalization of Van Vleck's matrix. It is the matrix that defines the form of the first quantum correction to the classical average of physical observables [7].

More explicitly, the equations of motion (5.32) following from variation of (5.33) read

$$\mathbf{T}'_a = \frac{\partial S}{\partial y^a} = T_a(x) + o(y) = 0\,, \qquad \mathbf{T}'_i = \frac{\partial S}{\partial x^i} = \partial_i T_a y^a + o(y^2) = 0\,. \tag{5.36}$$

As is seen, the dynamics on M do not decouple from those on the augmented configuration space $\mathcal{E}^*$ for arbitrary boundary conditions of y's, as opposite to (5.17). Nonetheless, imposing zero boundary conditions on y's, we can satisfy the second group of equations in (5.36) with $y = 0$ and x is arbitrary. Then the first group of equations reduces to the original equations of motion on M.

An important observation on the action (5.34) is that it has the form of local functional whenever the augmented constraints (5.23) are local.[10] Indeed, the only place where non-locality could emerge is the inversion of the augmented anchor (5.24). But, as is seen from (5.25), the inversion procedure, being performed perturbatively in y's, does not spoil locality. Therefore, the equivalent Lagrangian equations (5.32) are local and so is the action functional (5.33).

5.3 Quantizing non-Lagrangian dynamics via augmentation

As the augmented theory is always Lagrangian, its probability amplitude has the standard form

$$\Psi(x,y) = e^{\frac{i}{\hbar}S_\hbar(x,y)}\,, \qquad S_\hbar(x,y) = \sum_{n=0}^{\infty} \hbar^n S_n(x,y)\,. \tag{5.37}$$

Here the leading term $S_0(x,y)$ is given by the classical action (5.34) and the other terms can be regarded as quantum corrections to the naive path-integral measure $dxdy$ on $\mathcal{E}^*$. By definition (4.10), the probability amplitude (5.37) is a unique solution to the Schwinger-Dyson equations

$$\hat{\tilde{\mathbf{T}}}_I \Psi(x,y) = 0 \tag{5.38}$$

associated with the (over)complete set of augmented constraints (5.23). Notice that the $\phi\bar{\phi}$-symbols of the quantum constraint operators in (5.38) may defer from (5.23) by some quantum corrections in $\hbar$. These corrections can be systematically derived by solving the quantum master equation $\hat{\Omega}^2_{\mathrm{aug}} = 0$ for the augmented BRST operator.

What we are going to show in this section is that integrating the amplitude (5.37) of y's, we get the solution to the original Schwinger-Dyson equations (4.10). In other words, averaging the augmented probability amplitude (5.37) over the fibers of the vector bundle

[10] i.e., given by ordinary functions of fields $(\phi^I, \bar{\phi}_J)$ and their derivatives up to some finite order.

$\mathcal{E}^* \to M$ yields the probability amplitude for the original (non-)Lagrangian dynamics on M. We prove this statement under the following technical assumptions:

a) The normal symbol of the augmented BRST operator $\hat{\Omega}_{\rm aug}$, which may differ from the classical BRST charge (5.5) by some quantum corrections, still obeys Rel. (5.16).

b) Both the augmented and original constraint operators are hermitian (w.r.t. the standard inner product associated with the translation-invariant integration measures $dxdy$ and dx on $\mathcal{E}^*$ and M, respectively).

Note that the second condition follows from hermiticity requirement for the BRST operator provided that the fields $\bar{\eta}^I$ are chosen to be real, i.e., $(\bar{\eta}^I)^* = \bar{\eta}^I$.

Proposition 4. *Under the assumptions above,*

(i) the physical observables of the original theory are also observables of the augmented theory;

(ii)the functional

$$\psi(x) = \int dy \Psi(x,y)\,, \tag{5.39}$$

where $\Psi(x,y)$ is the Feynman amplitude (5.37) and the integral is taken over all y's satisfying zero boundary conditions, obeys the original Schwinger-Dyson equations (4.10) in coordinate representation;

(iii) let $\mathcal{O}$ be the physical observable associated with an on-shell gauge invariant function $O \in C^\infty(M)$ of the original theory, then the quantum average (4.17) is given by

$$\langle \mathcal{O} \rangle = (const) \int dydx\, O(x)\Psi(x,y)\,. \tag{5.40}$$

Remark 3. *The integrals (5.39) and (5.40), as they stand, are well defined only for theories of type* $(0,0)$. *In presence of gauge symmetries and/or Noether identities one should treat the action $S_\hbar(x,y)$ within the usual BV quantization method. This implies extension of the augmented configuration space $\mathcal{E}^*$ by ghost fields and imposing gauge fixing conditions that effectively reduces integration (5.40) to the space of gauge orbits. It is a perfectly standard technology and we will not dwell on it here.*

Proof. Statement (iii) is an immediate consequence of (i) and (ii).

We start with proving (ii). By definition, the amplitude $\Psi(x,y)$ is the matter state annihilated by the operators of augmented constraints. In particular, it is annihilate by the constraint operators that are extensions to $\mathcal{E}^*$ of the original constraints (3.15). We have

$$\hat{\mathbf{\Theta}}_I \Psi(x,y) = 0\,, \tag{5.41}$$

where the constraints $\mathbf{\Theta}_I = (\widetilde{\mathbf{T}}_a, \widetilde{\mathbf{R}}_\alpha)$ are defined by Rels. (5.23) and (5.27). The condition (5.16), being imposed on the normal symbol of the augmented BRST charge, suggests the following structure for the $\phi\bar{\phi}$-symbols of the constraint operators:

$$\mathbf{\Theta}_I = \Theta_I(x,\bar{x}) + o(\bar{y}). \tag{5.42}$$

That is the difference $\mathbf{\Theta}_I - \Theta_I$ between the original and augmented constraints is not only at least first order in y's it is also at least first order in $\bar{y}$'s. Now multiplying (5.41) on an arbitrary function of x^i and integrating the result over $\mathcal{E}^*$, we get

$$\begin{aligned} 0 &= \textstyle\int dxdy\, \Phi(x)\hat{\mathbf{\Theta}}_I \Psi(x,y) = \int dxdy\, (\hat{\mathbf{\Theta}}_I^\dagger \Phi(x))\Psi(x,y) \\ &= \textstyle\int dx\, \hat{\Theta}_I^\dagger \Phi(x) \int dy\, \Psi(x,y) = \int dx\, \Phi(x)\hat{\Theta}_I \int dy\, \Psi(x,y)\,. \end{aligned} \tag{5.43}$$

Here we have used the hermiticity requirements

$$\hat{\mathbf{\Theta}}_I^\dagger = \hat{\mathbf{\Theta}}_I\,, \qquad \hat{\Theta}_I^\dagger = \hat{\Theta}_I\,, \tag{5.44}$$

and Rel. (5.42). Since the function $\Phi(x)$ is arbitrary, the identity (5.43) is equivalent to the desired one

$$\hat{\Theta}_I \int dy\, \Psi(x,y) = 0\,. \tag{5.45}$$

Now, let us prove (i). As we have already mentioned in section 3.4, the space of physical observables is canonically isomorphic to the space of on-shell invariant functions on configuration space modulo trivial ones. In particular, a function $O \in C^\infty(\mathcal{E}^*)$ gives rise to a BRST invariant function on $\mathcal{N}_{\text{aug}}$, i.e., an observable of the augmented theory, iff

$$\mathbf{R}_\alpha O = \mathbf{W}_\alpha^a \mathbf{T}_a + \mathbf{W}_\alpha^i \mathbf{T}_i\,, \qquad \mathbf{R}_A O = \mathbf{W}_A^a \mathbf{T}_a + \mathbf{W}_A^i \mathbf{T}_i \tag{5.46}$$

for some $\mathbf{W}$'s. Here the generators of the augmented gauge algebra may differ from (5.27) by quantum corrections. Due to Proposition 2 and our assumptions these generators have the following structure:

$$\mathbf{R}_\alpha = R_\alpha^i(x)\frac{\partial}{\partial x^i} + R_\alpha^a(x,y)\frac{\partial}{\partial y^a}\,, \qquad \mathbf{R}_A = R_A^a(x,y)\frac{\partial}{\partial y^a}\,. \tag{5.47}$$

It remains to observe that for a y-independent function $O(x)$, eqs. (5.46) reduce to the on-shell invariance condition (3.25),

$$R_\alpha O = W_\alpha^a T_a\,. \tag{5.48}$$

This completes the proof.

Example 5. *Consider a Lagrangian theory with action $S(x)$. In this case, the dynamics bundle coincides with the cotangent bundle T^*M of the space of all histories. For simplicity sake assume that T^*M admits a flat connection. Given the canonical anchor (2.16), the augmented constraints (5.23) on $TM \oplus T^*M \oplus T^*M$ read*

$$\widetilde{\mathbf{T}}_i = \partial_i S(x) + \bar{x}_i - \bar{y}_i\,, \qquad \widetilde{\mathbf{T}}'_i = \partial_i S(x+y) - \partial_i S(x) - \bar{x}_i\,. \tag{5.49}$$

The action of the augmented theory (5.34) takes the form

$$S(x,y) = S(x+y) - S(x) = y^i \partial_i S(x) + \frac{1}{2} y^i y^j \partial_i \partial_j S(x) + \cdots\,. \tag{5.50}$$

After normalization, the quantum average of a physical observable $O(x)$ coincides with its usual value

$$\langle O \rangle = \int_{TM} dxdy\, O(x) e^{\frac{i}{\hbar}S(x,y)} = (\text{const}) \int_{M} dx\, O(x) e^{-\frac{i}{\hbar}S(x)} . \tag{5.51}$$

Of course, in the presence of gauge symmetries both these integrals are to be understood as integrals over the space of gauge orbits rather than over TM or M.

In this Lagrangian case, it is quite natural to interpret the augmentation fields y^i as the variations of the original fields x^i and this interpretation is automatically consistent with the zero boundary conditions for y's.

6. Examples of quantizing non-Lagrangian field theories

In this section, we demonstrate by examples what the Lagrange anchor can look like in non-Lagrangian relativistic field theory and how the general formalism described in previous sections works in practice. As the examples we consider two illustrative non-Lagrangian models: Maxwell electrodynamics with monopoles and self-dual p-form fields.

The Maxwell equations are considered in terms of the strength tensor and, in this formulation, they are not Lagrangian even without magnetic currents. If a magnetic monopole was point-like and satisfied the Dirac quantization condition, the theory would admit an equivalent Lagrangian formulation in terms of vector potential. We consider generic magnetic and electric sources, so the theory does not have any Lagrangian reformulation, although it still has a nontrivial Lagrange anchor, that is sufficient for a consistent quantization of the model. The observation about the structure of the Lagrange anchor might also be instructive for other non-Lagrangian field theories formulated in terms of strength tensors.

Studying the second example, we reverse the order of exposing the quantization procedure as compared to that described in sections 4 and 5. Given the equations of motion for a non-Lagrangian field theory in d dimension and a compatible Lagrange anchor, the general method allows one to equivalently reformulate this theory as a topological Lagrangian field theory in $d+1$ dimensions, with the original dynamics being localized at the boundary of this $(d+1)$-dimensional space-time. This also allows for a reverse consideration: one can start with an appropriate $(d+1)$-dimensional, topological Lagrangian theory and then identify "original" d-dimensional field equations and a Lagrange anchor in the action of the topological theory. In practice, this can be an instructive scheme for identifying those non-Lagrangian models that admit Poincaré covariant Lagrange anchors. To exemplify this idea, we take the Chern-Simons theory in $4n+3$ dimensions and reinterpret it as resulting from some quantization of self-dual $(2n+1)$-form fields in $4n+2$ dimensions.

6.1 Maxwell electrodynamics with monopoles

Consider the Maxwell equations with electric and magnetic currents:

$$d^\dagger \widetilde{F} = I\,, \qquad d^\dagger F = J\,. \tag{6.1}$$

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Here F is the strength of the electromagnetic field, whose Hodge dual is denoted by $\widetilde{F} = *F$, J and I are the electric and magnetic currents, respectively, and $d^\dagger = *d*$ is the adjoint exterior differential. As a consequence of eqs. (6.1), the currents J and I are conserved,

$$d^\dagger I = 0\,, \qquad d^\dagger J = 0\,. \tag{6.2}$$

Clearly, Equations (6.1), as they stand, are not Lagrangian, even if we set $I = 0$. (The number of equations is less than the number of fields).

Let us introduce the source P which is canonically conjugate to the field F. With the field F, being a 2-form on the space-time manifold, the source P is a bivector field on the same manifold. The canonical symplectic structure on the cotangent bundle of the space of all histories is given by

$$\omega = \int \delta\widetilde{F} \wedge \delta P'\,, \tag{6.3}$$

where the 2-form P' is obtained from P by lowering the upper indices with the space-time metric.

Consider now the following set of first-class constraints on the phase space of fields and sources:

$$\begin{gathered} T^1 = d^\dagger \widetilde{F} - I \approx 0\,, \qquad T^2 = d^\dagger(F + P') - J \approx 0\,, \\ \{T^a, T^b\} = 0\,, \qquad a, b = 1, 2\,. \end{gathered} \tag{6.4}$$

These constraints are obtained from (6.1) by adding the momentum depending term $d^\dagger P'$ to the second group of equations. It is the term that defines the canonical Lagrange anchor for the Maxwell electrodynamics [7]. Observe that the anchor is regular but not complete (see Definition 2.3). The physical meaning of this incompleteness can be understood in the following way. Let $I = 0$, then the first group of equations (6.1) expresses the closedness condition for the strength form F. The absence of momentum contributions to the corresponding constraints $T^1 \approx 0$ implies that we consider these equations as being pure non-Lagrangian in the sense of Theorem 2.5. Hence, no quantum fluctuations violate the closedness condition $dF = 0$, that guarantees the existence of a local gauge potential $A = d^{-1}F$ both at classical and quantum levels.

Notice that the constraints (6.4) are linearly dependent, $d^\dagger T^a = 0$, while the classical equations of motion (6.1) are not gauge invariant. Thus, according to Definition 2.1, we have a theory of type $(0, 1)$.

Upon canonical quantization, the constraints (6.4) turn into the following Schwinger-Dyson operators:

$$\hat{T}^1 = d^\dagger \widetilde{F} - I\,, \qquad \hat{T}^2 = d^\dagger\left(F - i\hbar\frac{\delta}{\delta F'}\right) - J\,, \tag{6.5}$$

F' being the contravariant strength tensor of electromagnetic field. The corresponding Schwinger-Dyson equation for the probability amplitude

$$\hat{T}^a \Psi[F] = 0 \tag{6.6}$$

is satisfied by

$$\Psi[F] = \Delta[T^1] e^{\frac{i}{\hbar} S[F]} . \tag{6.7}$$

Here

$$S[F] = \int \frac{1}{2} G d\widetilde{F} \wedge *d\widetilde{F} - G\widetilde{F} \wedge dJ \,, \qquad \Delta[T^1] = \int DC\delta[T^1 - dC] \,, \tag{6.8}$$

C is an auxiliary 0-form, and G is the inverse of the Laplace operator $\Box = dd^\dagger + d^\dagger d$. One can easily see that the distribution $\Delta[T^1]$, considered as the functional of F, is supported at the points where $T^1[F] = 0$ so that $T^1 \Delta[T^1] = 0$. (A naive solution to the last equation, namely $\Delta[T^1] = \delta[T^1]$, is ill defined because of linear dependence of the constraints T^1.) Notice that the amplitude (6.7) is non-Feynman: it is a nearly everywhere vanishing distribution on the configuration space of fields rather than a smooth, complex-valued function with absolute value 1. This fact is a direct consequence of incompleteness of the Lagrange anchor discussed above.

Passing to the momentum representation, we get the generating functional of Green's functions

$$\begin{aligned} Z[P] &= \int DF \Psi[F] e^{\frac{i}{\hbar}\int P' \wedge \widetilde{F}} = e^{\frac{i}{\hbar} W[P]} \,, \\ W[P] &= \int \frac{1}{2} G d^\dagger P' \wedge * d^\dagger P' - \bar{F} \wedge * P' \,, \end{aligned} \tag{6.9}$$

where

$$\bar{F} = dGJ + *dGI \tag{6.10}$$

is the mean electromagnetic field produced by the sources I and J. As with any free theory, the mean field $\bar{F}$ satisfies the classical equations of motion (6.1). One can also see that the propagator $\langle F(x)F(x')\rangle$ for the field F coincides with corresponding expression $\langle dA(x)dA(x')\rangle$ in the Maxwell electrodynamics with action $S[A] = \frac{1}{2}\int dA \wedge *dA$.

The probability amplitude (6.7) can also be arrived at by applying the augmentation method. By definition, the augmentation fields are the sections of the bundle which is dual to the dynamics bundle of the theory. So we introduce the 1-forms B^a, $a = 1, 2$; the pairing between the equations of motion and augmentation fields is given by the integral $\int T^a \wedge *B_a$. Since the constraints (6.4) are linear in fields and momenta, the action $S[F, B]$ of the augmented theory is at most quadratic in F and B^a. Specializing the general formulas (5.34) and (5.35) to the case at hand, we find

$$S[F, B] = \int (d^\dagger \widetilde{F} - I) \wedge *B_1 + (d^\dagger F - J) \wedge *B_2 + \frac{1}{2} dB_2 \wedge *dB_2. \tag{6.11}$$

The Noether identities between the original equations of motion (6.1) give rise to the gauge invariance of the action (6.11):

$$\delta_\varepsilon B_a = d\varepsilon_a \,, \qquad a = 1, 2 \,. \tag{6.12}$$

We can fix this arbitrariness by imposing the Lorentz gauges $d^\dagger B^a = 0$ on the augmentation fields and adding these constraints to the action (6.11) with the Lagrange multipliers C^a.

Then the gauge-fixed action reads

$$S_{\text{gf}}[F,B,C] = S[F,B] + \int d^\dagger B_a \wedge *C^a \,. \tag{6.13}$$

According to Proposition 5.4, the (non-Feynman) probability amplitude (6.7) admits the following path-integral representation in terms of the local action (6.13):

$$\Psi[F] = (const) \int DBDC e^{\frac{i}{\hbar} S_{\text{gf}}[F,B,C]} \tag{6.14}$$

Of course, in the case under consideration, one can verify the last equality directly, either by calculating the Gauss integrals over B's and C's or substituting (6.14) into the Schwinger-Dyson equation (6.6) and differentiating under the integral sign.

Given the probability amplitude (6.7), the quantum average of a physical observable O is defined by the path integral

$$\langle O \rangle = \int DFO[F]\Psi[F] \,. \tag{6.15}$$

In case $I = 0$, one can solve the constraint $T^2 = d^\dagger \widetilde{F} \approx 0$ in terms of the gauge potential A obeying the Lorentz gauge-fixing condition,

$$F = dA \,, \qquad d^\dagger A = 0 \,, \tag{6.16}$$

and integrate the pre-exponential Δ-functional in (6.7) as

$$DCDF\Delta[d^\dagger \widetilde{F} + dC] \quad \rightarrow \quad DA\delta[d^\dagger A] \,. \tag{6.17}$$

Then the integral (6.15) takes the form

$$\langle O \rangle = \int DAO[F(A)]\Psi[A] \,, \tag{6.18}$$

where

$$\Psi[A] = \delta[d^\dagger A] \exp \frac{i}{\hbar} \int \frac{1}{2} dA \wedge *dA + A \wedge *J \tag{6.19}$$

is nothing but the usual probability amplitude for the electromagnetic field subject to the Lorentz gauge.

6.2 Self-dual p-form fields

It has long been known that the quantization of chiral bosons in $(4n+2)$-dimensional space-time is closely related with the quantization of Chern-Simons theory in the space-time with one more dimension [14]. Roughly speaking, a physical wave-function of Chern-Simons fields on a $(4n + 3)$-dimensional manifold $\mathcal{M}$ can be treated as a probability amplitude (or partition function) for the self-dual fields living on the boundary of $\mathcal{M}$. For a recent discussion of the relationship between self-dual fields and the Chern-Simons theory we refer the reader to [15]. Below we justify and reinterpret this *ad hoc* quantization technique within the general method of sections 4 and 5.

Our starting point is the Chern-Simons action for the $(2n+1)$-form field $F \in \Lambda^{2n+1}(\mathcal{M})$ on a $(4n+3)$-dimensional manifold $\mathcal{M}$

$$S = -\frac{1}{2}\int_{\mathcal{M}} F \wedge DF\,, \tag{6.20}$$

D being the exterior differential on $\mathcal{M}$. Assume that $\mathcal{M} = M \times I$, where $I = [0,1] \subset \mathbb{R}$ and M is a compact $(4n+2)$-dimensional manifold without boundary. Then $\partial\mathcal{M} = M \cup M$.

Using the product structure of the manifold $\mathcal{M}$, one can globally decompose the field F and the operator D as

$$F = H + B \wedge dt\,, \qquad D = dt \wedge \partial_t + d\,, \tag{6.21}$$

Here $H \in \Lambda^{2n+1}(M)$ and $B \in \Lambda^{2n}(M)$ are the one-parameter families of differential forms labelled by $t \in [0,1]$ and d is the exterior differential on M. In this notation, the action (6.20) takes a simple Hamiltonian form (3.17), if one identifies t with evolution parameter:

$$S = \int_I dt \int_M \left(\frac{1}{2}H \wedge \dot{H} - B \wedge dH\right)\,. \tag{6.22}$$

The fist term in (6.22) defines (and is defined by) a symplectic structure on $\Lambda^{2n+1}(M)$; the corresponding symplectic 2-form reads

$$\omega = \int_M \delta H \wedge \delta H\,. \tag{6.23}$$

The field B plays the role of the Lagrange multiplier to the first-class constraints

$$T = dH \approx 0\,. \tag{6.24}$$

It is convenient to treat T as a linear functional (de Rham's flux) on the space of $2n$-forms:

$$T[\alpha] = \int_M \alpha \wedge dH\,, \qquad \forall \alpha \in \Lambda^{2n}(M)\,. \tag{6.25}$$

Then one can easily check that the constraints (6.24) have vanishing Poisson brackets,

$$\{T[\alpha], T[\beta]\} = \int_M d\alpha \wedge d\beta = 0\,, \qquad \forall \alpha,\beta \in \Lambda^{2n}(M). \tag{6.26}$$

Since $d^2 = 0$, these constraints are reducible, $dT \equiv 0$, and one can further deduce that the order of reducibility is $2n$.

The Hamiltonian reduction by the first-class constraints (6.24) leads to a finite dimensional phase space. We have a rather explicit description of the reduced phase space due to the Hodge decomposition

$$\Lambda^{2n+1}(M) = d\Lambda^{2n}(M) \oplus d^\dagger \Lambda^{2n+2}(M) \oplus \Lambda^{2n+1}_H(M)\,. \tag{6.27}$$

Here $d^\dagger : \Lambda^m(M) \to \Lambda^{m-1}(M)$ is the adjoint differential constructed by some Riemannian metric on M, and $\Lambda^{2n+1}_H(M)$ is the subspace of harmonic forms. According to the Hodge theory, the space $\Lambda_H^{2n+1}(M)$ is naturally isomorphic to the de Rham cohomology group $H^{2n+1}(M)$. The first-class constraints (6.24) single out the coisotropic subspace of d-closed forms, whose complementary isotropic subspace is given by the $d^\dagger$-exact forms. Notice that both these subspaces are Lagrangian iff $H^{2n+1}(M) = 0$.

The Hamiltonian flux generated by the first-class constraints changes any $(2n+1)$-form H by an exact one:

$$\delta_\varepsilon H = \{H, T[\varepsilon]\} = d\varepsilon\,. \tag{6.28}$$

Taking the quotient of d-closed $(2n+1)$-forms by d-exact ones, we obtain the physical phase space of the model, which is apparently isomorphic to the (finite dimensional) subspace of harmonic forms on M:

$$\Lambda^{2n+1}(M)//T \simeq \Lambda_H^{2n+1}(M) \simeq H^{2n+1}(M)\,. \tag{6.29}$$

We are lead to conclude that the model under consideration is not topological unless $H^{2n+1}(M) = 0$. (We define a topological theory as a theory without physical degrees of freedom.) To get rid of the physical modes and obtain a pure topological model we can restrict the dynamics on the affine subspace $\Lambda_\alpha^{2n+1}(M) \subset \Lambda^{2n+1}(M)$ constituted by the forms $H = H_\alpha + H_0$, where $H_0 \in d\Lambda^{2n}(M) \oplus d^\dagger\Lambda^{2n+2}(M)$ and H_α is a time-independent harmonic $(2n+1)$-form representing the de Rham class $\alpha = [H_\alpha] \in H^{2n+1}(M)$. This restriction is compatible with dynamics. Indeed, the equations of motion following from the Hamiltonian action (6.22) read

$$\dot{H} = dB\,, \qquad dH = 0\,. \tag{6.30}$$

So, the de Rham class $[H]$ of the closed form H does not change with time and can thus be regarded as a (topological) integral of motion. Moreover, the embedding $\Lambda_\alpha^{2n+1}(M) \subset \Lambda^{2n+1}(M)$ is symplectic, i.e., the restriction of the 2-form (6.23) to $\Lambda_\alpha^{2n+1}(M)$ is nondegenerate.

To further proceed with the interpretation and quantization of the Chern-Simons theory on the product manifold $\mathcal{M} = M \times I$, let us endow M with a Lorentzian metric. (A necessary and sufficient condition for such a metric to exist is that the Euler characteristic $\chi(M)$ be zero.) Then, the corresponding Hodge operator $*$ squares to $+1$ on the middle forms so that any $(2n+1)$-form H admits a unique decomposition in the sum of its self-dual and anti-self-dual parts:

$$H = H^+ + H^-\,, \qquad *H^\pm = \pm H^\pm\,. \tag{6.31}$$

Since $\omega(\delta H^\pm, \delta H^\pm) = 0$, we have a natural polarization of the phase space $\Lambda^{2n+1}(M)$ given by the two complementary Lagrangian subspaces of self- and anti-self-dual forms:

$$\Lambda^{2n+1}(M) = \Lambda_+^{2n+1}(M) \oplus \Lambda_-^{2n+1}(M)\,. \tag{6.32}$$

Let us regard the fields H^- as the “momentum coordinates” canonically conjugate to the “position coordinates” H^+ and rewrite the Hamiltonian action (6.22) as

$$S = \int_I dt \int_M H^- \wedge \dot{H}^+ - B \wedge d(H^+ + H^-) \tag{6.33}$$

Upon restriction to $\Lambda^{2n+1}_\alpha(M)$ this action describes a topological field theory and its form is identical to the form of the topological action (3.17). Recall that the latter was constructed on general grounds starting from some classical (not necessarily Lagrangian) equations of motion supplemented with an appropriate Lagrange structure. Proceeding now backward, we can readily reinterpret the topological model (6.33) in terms of non-Lagrangian dynamics on M.

Namely, the classical equations of motion are to be identified with the momentum independent terms in the Hamiltonian constraints

$$T = d(H^+ + H^-) \approx 0\,. \tag{6.34}$$

With our choice of the phase-space polarization this yields the closedness condition for the self-dual form H^+,

$$dH^+ = 0\,. \tag{6.35}$$

Similar to the Hamiltonian constraints (6.34), these equations are $2n$-times reducible, and hence they define a regular gauge theory of type $(0, 2n)$ (although there is no gauge invariance in the usual sense). A non-Lagrangian nature of equations (6.35) was discussed at length in [13].

The second term in the Hamiltonian constraints (6.34), namely dH^-, is linear in momenta and should be identified with the Lagrange anchor. Zero boundary conditions (3.19) on the momenta

$$H^-|_{t=0,1} = 0 \tag{6.36}$$

ensure the equivalence of the classical dynamics (6.30) and (6.35), where both H and H^+ belong to $\Lambda^{2n+1}_\alpha(M)$. Fixing the de Rham class $[H^+] = \alpha$ of a solution to equation (6.35) is quite similar to fixing the boundary conditions for a field theory on a bounded space-time domain.

Quantizing now the Hamiltonian constraints (6.34) in the coordinate representation, we get the following Schwinger-Dyson operator:

$$\hat{T} = d\left(H^+ - i\hbar\frac{\delta}{\delta H'^+}\right)\,. \tag{6.37}$$

Here H'^+ is the bivector on M obtained from H^+ by rising indices with the help of the Lorentzian metric. The probability amplitude on the configuration space $\Lambda^{2n+1}_\alpha(M) \cap \Lambda^{2n+1}_+(M)$ is determined by the equation

$$\hat{T}\Psi[H^+] = 0\,. \tag{6.38}$$

We use the augmentation method to write down an explicit path-integral representation for $\Psi[H^+]$. To this end, we introduce the augmentation fields $C \in \Lambda^{2n}(M)$ whose configuration space is dual to the linear space $\Lambda^{2n+2}(M)$ of the field equations (6.35), and apply the general formulas (5.34), (5.35), and (5.37) to construct the Feynman probability amplitude of the augmented theory. The result is almost obvious:

$$\Psi[C,H^+] = e^{\frac{i}{\hbar}S[C,H^+]}\,,$$
$$S[C,H^+] = \int -\frac{1}{2}dC\wedge *dC + H^+\wedge dC\,. \tag{6.39}$$

The corresponding equations of motion read

$$dH^+ = 0\,, \qquad (dC)^- = 0\,. \tag{6.40}$$

The augmented theory is seen to describe the pair of self-dual fields: one in terms of the "strength tensor" H^+ and another one in terms of the "gauge potential" C. Notice that the Noether identities for the non-Lagrangian equations of motion (6.35), i.e., $d(dH^+)\equiv 0$, reincarnate as the gauge transformations of the augmentation fields:

$$C\to C' = C + dA\,, \qquad \forall A\in\Lambda^{2n-1}(M)\,. \tag{6.41}$$

Integrating formally the amplitude (6.39) over the fields C, we obtain the probability amplitude for the self-dual field H^+,

$$\Psi[H^+] = \int DC\Psi[C,H^+]\,. \tag{6.42}$$

It is now just a matter of differentiating under the integral sign to show that the amplitude (6.42) does obey the Schwinger-Dyson equation (6.38). We have

$$\hat{T}\Psi[H^+] = \int DC\hat{T}e^{\frac{i}{\hbar}S[C,H^+]} = i\hbar * \int DC\frac{\delta}{\delta C}e^{\frac{i}{\hbar}S[C,H^+]} = 0\,. \tag{6.43}$$

A more rigor treatment of the (divergent) Gaussian integral (6.42) implies fixing the gauge freedom (6.41) by the BV method for reducible gauge-algebra generators [5, 6].

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